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Inheritance of hyper-duality in imprimitive Bose–Mesner algebras

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Abstract

We prove the following result concerning the inheritance of hyper-duality by block and quotient Bose–Mesner algebras associated with a hyper-dual pair of imprimitive Bose–Mesner algebras. Let $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ denote Bose–Mesner algebras. Suppose there is a hyper-duality ψ from the subconstituent algebra of $\mathscr{M}$ with respect to p to the subconstituent algebra of $\widetilde{\mathscr{M}}$ with respect to $\widetilde{p}$. Also suppose that $\mathscr{M}$ is imprimitive with respect to a subset $\mathscr{I}$ of Hadamard idempotents, so $\widetilde{\mathscr{M}}$ is dual imprimitive with respect to the subset $\Psi(\mathscr{I})$ of primitive idempotents, where $\Psi : \mathscr{M} \to \widetilde{\mathscr{M}}$ is the formal duality associated with ψ. Let $\mathscr{B}$ denote the block Bose–Mesner algebra of $\mathscr{M}$ on the block containing p, and let $\widetilde{\mathscr{Q}}$ denote the quotient Bose–Mesner algebra of $\widetilde{\mathscr{M}}$ with respect to $\Psi(\mathscr{I})$. Then there is a hyper-duality from the subconstituent algebra of $\mathscr{B}$ with respect to p to the subconstituent algebra of $\widetilde{\mathscr{Q}}$ with respect to $\widetilde{p}$.

Keywords: Block Bose–Mesner algebra; Quotient Bose–Mesner algebra; Subconstituent algebra; Terwilliger algebra

1. Introduction

We study the inheritance of hyper-duality by block and quotient Bose–Mesner algebras associated with imprimitive hyper-dual pairs of Bose–Mesner algebras. Let $\mathscr{M}$ denote a Bose–Mesner algebra. Then $\mathscr{M}$ is imprimitive with respect to some subset $\mathscr{I}$ of its Hadamard idempotents whenever $\mathscr{I}$ spans a proper nonzero subalgebra of $\mathscr{M}$, and $\mathscr{M}$ is dual imprimitive with respect to some subset $\mathscr{D}$ of its primitive idempotents whenever $\mathscr{D}$ spans a proper nonzero subalgebra with respect to entry-wise multiplication of $\mathscr{M}$. If $\mathscr{M}$ is imprimitive with respect to $\mathscr{I}$, then restricting the Hadamard idempotents in $\mathscr{I}$ to any one of certain subsets (blocks) of the underlying point set yields the Hadamard idempotents of a block Bose–Mesner algebra $\mathscr{B}$. If $\mathscr{M}$ is dual imprimitive with respect to $\mathscr{D}$, then restricting the primitive idempotents in $\mathscr{D}$ to certain subsets of the underlying point set (class representatives for a particular equivalence relation) and renormalizing yields the primitive idempotents of a quotient Bose–Mesner algebra $\mathscr{Q}$.

In the presence of a formal duality, the dual nature of these constructions yields the following. Suppose $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ are Bose–Mesner algebras, and suppose $\Psi : \mathscr{M} \to \widetilde{\mathscr{M}}$ is a formal duality. If $\mathscr{M}$ is imprimitive with respect to $\mathscr{I}$, then $\widetilde{\mathscr{M}}$ is dual imprimitive with respect to $\Psi(\mathscr{I})$. Moreover, each block Bose–Mesner algebra $\mathscr{B}$ of $\mathscr{M}$ with respect to $\mathscr{I}$ is formally dual to the quotient Bose–Mesner algebra $\widetilde{\mathscr{Q}}$ of $\widetilde{\mathscr{M}}$ with respect to $\Psi(\mathscr{I})$.

In this paper we extend the above fact to hyper-duality for Bose–Mesner algebras [6,10]. Hyper-duality is an extension of formal duality of Bose–Mesner algebras to their subconstituent (Terwilliger) algebras with respect to some base points. Hyper-duality implies formal duality; indeed, associated with each hyper-duality is a unique formal duality

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for the underlying Bose–Mesner algebras. Our main result is the following. Suppose there is a hyper-duality ψ from the subconstituent algebra of $\mathscr{M}$ with respect to p to the subconstituent algebra of $\widetilde{\mathscr{M}}$ with respect to $\widetilde{p}$. Also suppose that $\mathscr{M}$ is imprimitive with respect to $\mathscr{I}$, so $\widetilde{\mathscr{M}}$ is dual imprimitive with respect to $\Psi(\mathscr{I})$, where $\Psi : \mathscr{M} \to \widetilde{\mathscr{M}}$ is the formal duality associated with ψ. Let $\mathscr{B}$ denote the block Bose–Mesner algebra of $\mathscr{M}$ on the block containing p, and let $\widetilde{\mathscr{Q}}$ denote the quotient Bose–Mesner algebra of $\widetilde{\mathscr{M}}$ with respect to $\Psi(\mathscr{I})$. Then there is a hyper-duality from the subconstituent algebra of $\mathscr{B}$ with respect to p to the subconstituent algebra of $\widetilde{\mathscr{Q}}$ with respect to $\widetilde{p}$.

This paper is organized as follows. In Section 2 we recall Bose–Mesner and subconstituent algebras. In Sections 3–5 we recall imprimitivity and the constructions of block and quotient Bose–Mesner algebras from an imprimitive Bose–Mesner algebra. In Section 6, we recall formal and hyper-duality for pairs of Bose–Mesner algebras. Most of the material in Sections 2–6 is generally well known. In Section 7 we prove our main results. In Section 8 we specialize the main results to the case of self-duality. Finally, in Section 9 we apply our results to the folded Hamming cubes and the halved Hamming cubes. The reader may find it illustrative to work out the details of this example while going through the paper, although it assumes some knowledge of P- and Q-polynomial association schemes.

2. Bose–Mesner and subconstituent algebras

In this section we review Bose–Mesner and subconstituent algebras. The reader is referred to [2,3] for more details concerning Bose–Mesner algebras.

For any finite, nonempty set X, write $\mathbb{M}_X$ to denote the complex algebra of matrices with complex entries whose rows and columns are indexed by X. For $A \in \mathbb{M}_X$ and for $x, y \in X$, let $A(x, y)$ denote the (x, y)-entry of A. For A, $B \in \mathbb{M}_X$, let $A \circ B$ denote the Hadamard (entry-wise, Schur) product of A and B: $(A \circ B)(x, y) = A(x, y)B(x, y)$ for all $x, y \in X$. Denote the ordinary matrix product of A and B by juxtaposition: AB. For $A \in \mathbb{M}_X$, let ${}^{t}A$ and $\overline{A}$ denote the transpose and the complex conjugate, respectively.

A *Bose–Mesner algebra* on X is a commutative subalgebra $\mathscr{M}$ of $\mathbb{M}_X$ which is closed under Hadamard product, which is closed under transposition, and which contains the identity matrix I and the all-ones matrix J. Let $\mathscr{M}$ denote a $(d + 1)$-dimensional Bose–Mesner algebra on X. Then $\mathscr{M}$ has a unique basis $\{A_i\}_{i=0}^{d}$ such that

$$A_0 = I, \quad A_i \circ A_j = \delta_{ij} A_i \quad (0 \leqslant i, j \leqslant d), \quad \sum_{i=0}^{d} A_i = J, \tag{1}$$

where δ_{ij} denotes the Kronecker symbol. We call $\{A_i\}_{i=0}^{d}$ the *basis of Hadamard idempotents* of $\mathscr{M}$. Observe that each A_i has entries in $\{0, 1\}$ since $A_i \circ A_i = A_i$. In addition, $\mathscr{M}$ has a unique basis $\{E_i\}_{i=0}^{d}$ such that

$$E_0 = |X|^{-1} J, \quad E_i E_j = \delta_{ij} E_i \quad (0 \leqslant i, j \leqslant d), \quad \sum_{i=0}^{d} E_i = I. \tag{2}$$

We call $\{E_i\}_{i=0}^{d}$ the *basis of primitive idempotents* of $\mathscr{M}$. It is known that ${}^{t}E_i = \overline{E_i}$ $(0 \leqslant i \leqslant d)$. Throughout this paper we shall assume that the Hadamard and primitive idempotents of any given Bose–Mesner algebra are ordered.

Let $\mathscr{M}$ denote a $(d + 1)$-dimensional Bose–Mesner algebra on X. Let $\{A_i\}_{i=0}^{d}$ and $\{E_i\}_{i=0}^{d}$ denote orderings of the Hadamard and primitive idempotents of $\mathscr{M}$. Then for $0 \leqslant h, i, j \leqslant d$, the *intersection numbers* p_{ij}^{h}, the *Krein parameters* q_{ij}^{h}, the *eigenvalues* $P(j, i)$, and the *dual eigenvalues* $Q(j, i)$ of $\mathscr{M}$ are defined, respectively, by

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^{h} A_h, \quad E_i \circ E_j = |X|^{-1} \sum_{h=0}^{d} q_{ij}^{h} E_h \quad (0 \leqslant i, j \leqslant d), \tag{3}$$

$$A_i = \sum_{j=0}^{d} P(j, i) E_j, \quad E_i = |X|^{-1} \sum_{j=0}^{d} Q(j, i) A_j \quad (0 \leqslant i \leqslant d). \tag{4}$$

The intersection numbers are nonnegative integers, and the Krein parameters are nonnegative real numbers. The $(d + 1) \times (d + 1)$ matrices P and Q with (j, i)-entries $P(j, i)$ and $Q(j, i)$ are called the *eigenmatrix* and *dual* *eigenmatrix* of $\mathscr{M}$, respectively. The *valency* of A_i is the number $k_i := P(0, i)$. It is well known that k_i is equal to the number of ones in each row of A_i. The *multiplicity* of E_i is the number $m_i := Q(0, i)$. It is well known that m_i is equal to the rank and trace of E_i.

Each of these four sets of parameters (intersection numbers, Krein parameters, eigenvalues, and dual eigenvalues) determines the other three [2]. These parameters carry all of the algebraic structure of a Bose–Mesner algebra. That is to say, any two Bose–Mesner algebras with the same parameters are isomorphic as Bose–Mesner algebras. It is well known that the Hadamard idempotents of a Bose–Mesner algebra encode the combinatorial structure of an association scheme. There are many nonisomorphic association schemes that have isomorphic Bose–Mesner algebras.

We recall some formulas [2,3] concerning the intersection numbers and Krein parameters which we shall need later. First, note that transposition defines involutions on the Hadamard and primitive idempotents: Let i^{t} and i_t be such that $A_{i^{\mathrm{t}}}={}^{\mathrm{t}}A_i$ and $E_{i_t}={}^{\mathrm{t}}E_i$. With this notation,

$$k_h p^h_{ij} = k_j p^j_{i^{\mathrm{t}}h} = k_i p^i_{hj^{\mathrm{t}}}, \quad m_h q^h_{ij} = m_j q^j_{i_t h} = m_i q^i_{hj_t} \quad (0 \leqslant h, i, j \leqslant d), \tag{5}$$

$$k_i = p^0_{i^{\mathrm{t}}i}, \quad m_i = q^0_{i_t i} \quad (0 \leqslant i \leqslant d), \tag{6}$$

$$\sum_{i=0}^{d} p^h_{ij} = k_j, \quad \sum_{i=0}^{d} q^h_{ij} = m_j \quad (0 \leqslant h, j \leqslant d). \tag{7}$$

We now recall the subconstituent algebra of a Bose–Mesner algebra. See [24] for further details. Let $\mathscr{M}$ be a $(d+1)$-dimensional Bose–Mesner algebra on X. Fix a "base point" $p \in X$. For $A \in \mathscr{M}$, let $\rho(A) \in \mathbb{M}_X$ denote the diagonal matrix with (x, x)-entry $\rho(A)(x, x) = A(p, x)$. Let $\mathscr{M}^* = \{\rho(A) \mid A \in \mathscr{M}\}$. Then $\mathscr{M}^*$ is called the *dual Bose–Mesner algebra* of $\mathscr{M}$ (with respect to p). The map $\rho : \mathscr{M} \to \mathscr{M}^*$ is a linear bijection. Set $E^*_i = \rho(A_i)$ and $A^*_i = \rho(|X|E_i)$ $(0 \leqslant i \leqslant d)$. Since ρ is a linear bijection, $\{E^*_i\}^d_{i=0}$ and $\{A^*_i\}^d_{i=0}$ are bases of $\mathscr{M}^*$. We refer to $\{E^*_i\}^d_{i=0}$ and $\{A^*_i\}^d_{i=0}$ as the *basis of dual idempotents* and the *basis of dual Hadamard idempotents* of $\mathscr{M}^*$, respectively. The dual and dual Hadamard idempotents inherit the orderings of the Hadamard and primitive idempotents of $\mathscr{M}$, respectively. The subalgebra $\mathscr{T} = \mathscr{T}(p)$ of $\mathbb{M}_X$ generated by $\mathscr{M} \cup \mathscr{M}^*$ is called the *subconstituent* (or *Terwilliger*) *algebra* of $\mathscr{M}$ (with respect to p). The use of * in connection with subconstituent algebras is not related to the operation of taking the conjugate-transpose of a matrix.

3. Imprimitive Bose–Mesner algebras

In this section we review some material concerning imprimitive Bose–Mesner algebras. The reader is referred to [2,3] for summaries of imprimitivity. We note the following concerning our treatment of imprimitivity. To describe the associated subconstituent algebras, we need to carefully track a base point as we pass from an imprimitive Bose–Mesner algebra to the appropriate block Bose–Mesner algebra and the quotient Bose–Mesner algebra. Thus we avoid the common practices of reordering the point set and canonical bases [2] and of expressing certain elements of the imprimitive Bose–Mesner algebra as a Kronecker product of two matrices [2,3]. As a bridge from Bose–Mesner to subconstituent algebras, we formulate some of these results as statements concerning elements of the Bose–Mesner algebra whereas most references only state the equivalent results concerning intersection numbers and Krein parameters [3,21]. Finally, we note that our treatment of imprimitivity is by no means comprehensive. For example, the references [2–4,17,21,25] treat various aspects of imprimitivity which we do not discuss here.

We present brief proofs of some facts concerning imprimitivity. In doing so, we favor proofs which can be "dualized" in the sense which Bannai and Ito [2] have dubbed "Kawada–Delsarte duality" [11]. Given a statement concerning Bose–Mesner algebras, its Kawada–Delsarte dual is formed by swapping each instance of ordinary and Hadamard multiplication, of I and J (taking care to distinguish them from A_0 and $|X|E_0$), of A_i and E_i, of p^h_{ij} and $q^h_{ij}/|X|$, and of $P(j, i)$ and $Q(i, j)/|X|$. Kawada–Delsarte duality formalizes the fact that the Kawada–Delsarte dual of any "purely algebraic" property of all Bose–Mesner algebras is also a property of all Bose–Mesner algebras (this is not true of certain "combinatorial" properties or necessarily of all properties of any particular example). A number of properties and their Kawada–Delsarte duals are stated in the previous section, and these are all that will be needed to "dualize" the proofs below.

Lemma 3.1. *Let $\mathscr{M}$ denote a $(d+1)$-dimensional Bose–Mesner algebra on X. Let $\mathscr{I}$ denote a subset of* $\{0, 1, \ldots, d\}$. *Then the following are equivalent*:

(i) *The subset $\{A_i\}_{i\in\mathscr{I}}$ of the Hadamard idempotents of $\mathscr{M}$ spans a matrix subalgebra with respect to the ordinary matrix product.*
(ii) $A_i A_j = \sum_{h\in\mathscr{I}} p_{ij}^h A_h$ *for all* $i, j \in \mathscr{I}$.
(iii) $p_{ij}^k = 0$ *when* $i, j \in \mathscr{I}$ *and* $k \notin \mathscr{I}$.

$\mathscr{M}$ *is said to be* imprimitive with respect to $\mathscr{I}$ *whenever* (i)–(iii) *hold and $\mathscr{I}$ is neither* $\{0\}$ *nor* $\{0, 1, \ldots, d\}$. *Moreover, if* (i)–(iii) *hold, then the following hold.*

(a) ${}^{\mathrm{t}}A_j \in \{A_h\}_{h\in\mathscr{I}}$ *for all* $j \in \mathscr{I}$ *(i.e.* $j \in \mathscr{I}$ *implies* $j^t \in \mathscr{I}$).
(b) $I = A_0 \in \{A_h\}_{h\in\mathscr{I}}$ *(i.e.* $0 \in \mathscr{I}$).
(c) *There exists a unique subset $\mathscr{D}$ of* $\{0, 1, \ldots, d\}$ *such that*

$$\sum_{i\in\mathscr{I}} A_i = k_{\mathscr{I}} \sum_{i\in\mathscr{D}} E_i,$$

where $k_{\mathscr{I}} = \sum_{i\in\mathscr{I}} k_i$.

Proof. The equivalence of (i)–(iii) is clear from (3).

(a) Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ denote the distinct entries in column i of the eigenmatrix P of $\mathscr{M}$. Then $A_i = \sum_{\ell=1}^r \lambda_\ell \sum_{\{j|P_{ji}=\lambda_\ell\}} E_j$ by (4), and ${}^{\mathrm{t}}A_i = {}^{\mathrm{t}}\bar{A_i} = \sum_{\ell=1}^r \bar{\lambda}_\ell \sum_{\{j|P_{ji}=\lambda_\ell\}} E_j$. The E_j are idempotents, so $A_i^k = \sum_{\ell=1}^r \lambda_\ell^k \sum_{\{j|P_{ji}=\lambda_\ell\}} E_j$. The λ_ℓ are distinct, so each sum $\sum_{\{j|P_{ji}=\lambda_\ell\}} E_j$ lies in the span of $A_i, A_i^2, A_i^3, \ldots, A_i^r$ since the coefficient matrix is essentially Vandermonde. The result follows from (i).
(b) Observe that $p_{i^t i}^0 = k_i > 0$ by (6), so $0 \in \mathscr{I}$ by (iii) and (a).
(c) Set $S = k_{\mathscr{I}}^{-1} \sum_{i\in\mathscr{I}} A_i$. By (3) and (ii), $S^2 = k_{\mathscr{I}}^{-2} \sum_{i,j\in\mathscr{I}} \sum_{h=0}^d p_{ij}^h A_h = k_{\mathscr{I}}^{-2} \sum_{h,i,j\in\mathscr{I}} p_{ij}^h A_h$. Note that $p_{ij}^h = 0$ if and only if $p_{i^t h}^j = 0$ by (5). Thus by (a), (7), and (iii), $S^2 = k_{\mathscr{I}}^{-2} \sum_{i,h\in\mathscr{I}} \sum_{j=0}^d p_{ij}^h A_h = k_{\mathscr{I}}^{-2} \sum_{i\in\mathscr{I}} k_i \sum_{h\in\mathscr{I}} A_h = S$. Since S is an idempotent, it must be a sum of primitive idempotents, $S = \sum_{i\in\mathscr{D}} E_i$ for some subset $\mathscr{D}$ of $\{0, 1, \ldots, d\}$, as required. □

Lemma 3.2. *Let $\mathscr{M}$ denote a $(d+1)$-dimensional Bose–Mesner algebra on X. Let $\mathscr{D}$ denote a subset of* $\{0, 1, \ldots, d\}$. *Then the following are equivalent*:

(i) *The subset $\{E_i\}_{i\in\mathscr{D}}$ of the primitive idempotents of $\mathscr{M}$ spans a matrix subalgebra with respect to the Hadamard product.*
(ii) $E_i \circ E_j = |X|^{-1} \sum_{h\in\mathscr{D}} q_{ij}^h E_h$ *for all* $i, j \in \mathscr{D}$.
(iii) $q_{ij}^k = 0$ *when* $i, j \in \mathscr{D}$ *and* $k \notin \mathscr{D}$.

$\mathscr{M}$ *is said to be* dual imprimitive with respect to $\mathscr{D}$ *whenever* (i)–(iii) *hold and $\mathscr{D}$ is neither* $\{0\}$ *nor* $\{0, 1, \ldots, d\}$. *Moreover, if* (i)–(iii) *hold, then the following hold.*

(a) ${}^{\mathrm{t}}E_j \in \{E_h\}_{h\in\mathscr{D}}$ *for all* $j \in \mathscr{D}$ *(i.e.* $j \in \mathscr{D}$ *implies* $j_t \in \mathscr{D}$).
(b) $|X|^{-1} J = E_0 \in \{E_h\}_{h\in\mathscr{D}}$ *(i.e.* $0 \in \mathscr{D}$).
(c) *There exists a unique subset $\mathscr{I}$ of* $\{0, 1, \ldots, d\}$ *such that*

$$|X| \sum_{i\in\mathscr{D}} E_i = m_{\mathscr{D}} \sum_{i\in\mathscr{I}} A_i,$$

where $m_{\mathscr{D}} = \sum_{i\in\mathscr{D}} m_i$.

Proof. Use the proof of Lemma 3.1 with the Kawada–Delsarte dual of each statement. □

Lemma 3.3. *Let $\mathscr{M}$ denote a $(d+1)$-dimensional Bose–Mesner algebra on X.*

(i) *If $\mathscr{M}$ is imprimitive with respect to $\mathscr{I}$, then $\mathscr{M}$ is dual imprimitive with respect to the set $\mathscr{D}$ of Lemma* 3.1(c).
(ii) *If $\mathscr{M}$ is dual imprimitive with respect to $\mathscr{D}$, then $\mathscr{M}$ is imprimitive with respect to the set $\mathscr{I}$ of Lemma* 3.2(c).

As imprimitivity and dual imprimitivity coincide, we say in the above cases that $\mathscr{M}$ is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$.

Proof. (i) Note that $\sum_{h\in\mathscr{I}} A_h = k_{\mathscr{I}} \sum_{i\in\mathscr{D}} E_i$ is an idempotent for Hadamard multiplication, so

$$k_{\mathscr{I}} \sum_{i\in\mathscr{D}} E_i = \left(k_{\mathscr{I}} \sum_{i\in\mathscr{D}} E_i\right)^{\circ 2} = k_{\mathscr{I}}^2 \sum_{i,j\in\mathscr{D}} E_i \circ E_j = k_{\mathscr{I}}^2 \sum_{h=0}^{d} \left(|X|^{-1} \sum_{i,j\in\mathscr{D}} q_{ij}^h\right) E_h.$$

Since the primitive idempotents are linearly independent, $\sum_{i,j\in\mathscr{D}} q_{ij}^h = 0$ for $h \notin \mathscr{D}$. But the Krein parameters are nonnegative real numbers, so $q_{ij}^h = 0$ when $i, j \in \mathscr{D}$ and $h \notin \mathscr{D}$. Thus $\mathscr{M}$ is dual imprimitive with respect to $\mathscr{D}$.

(ii) Use the proof of (i) with the Kawada–Delsarte dual of each statement. □

Lemma 3.4. *Suppose $\mathscr{M}$ is a $(d+1)$-dimensional Bose–Mesner algebra on X which is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$. Then $k_{\mathscr{I}} m_{\mathscr{D}} = |X|$.*

Proof. Combine Lemmas 3.1(c) and 3.2(c). □

In the next two sections we recall the constructions of block and quotient Bose–Mesner algebras from imprimitive Bose–Mesner algebras. We shall use some nonstandard notation, namely $\llbracket\ \rrbracket$ and $\ulcorner\!\!\ulcorner\ \urcorner\!\!\urcorner$ for the respective objects associated with a block and a quotient Bose–Mesner algebra. As a mnemonic, we suggest the similarity in shape between the letter "b" for block and the left side of $\llbracket\ \rrbracket$ and the similarity in shape between the letter "q" for quotient and the right side of $\ulcorner\!\!\ulcorner\ \urcorner\!\!\urcorner$. We also find $\llbracket\ \rrbracket$ and $\ulcorner\!\!\ulcorner\ \urcorner\!\!\urcorner$ suggestive of the dual nature of block and quotient Bose–Mesner algebras which we are stressing in this paper.

4. Block Bose–Mesner algebras

Given an imprimitive Bose–Mesner algebra, one may construct related block Bose–Mesner algebras. To describe them we define equivalence relations on X and on $\{0, 1, \ldots, d\}$.

Lemma 4.1. *Suppose $\mathscr{M}$ is a $(d+1)$-dimensional Bose–Mesner algebra on X which is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$. Then the following are equivalence relations.*

(i) *The relation $\sim$ on X defined by $x \sim y$ if and only if $(\sum_{i\in\mathscr{I}} A_i)(x, y) = 1$. Let $\llbracket X\rrbracket$ denote the set of $\sim$ equivalence classes, and write $\llbracket x\rrbracket$ to denote the $\sim$ equivalence class of $x \in X$.*
(ii) *The relation $\simeq$ on $\{0, 1, \ldots, d\}$ defined by $i \simeq j$ if and only if $q_{ih}^j \neq 0$ for some $h \in \mathscr{D}$. Let $\llbracket\mathscr{D}\rrbracket$ denote the set of $\simeq$ equivalence classes, and write $\llbracket i\rrbracket$ to denote the $\simeq$ equivalence class of $i \in \{0, 1, \ldots, d\}$.*

Proof. (i) Observe that $\sim$ is reflexive by Lemma 3.1(b) and symmetric by Lemma 3.1(a). Suppose $x \sim y$, $y \sim z$. Then there exist $i, j \in \mathscr{I}$ such that $A_i(x, y) = 1$, $A_j(y, z) = 1$. Hence the (x, z)-entry of $A_i A_j$ is nonzero. Since $\{A_k\}_{k\in\mathscr{I}}$ spans a matrix algebra, there exists $k \in \mathscr{I}$ such that $A_k(x, z) = 1$. Thus $x \sim z$, so $\sim$ is transitive.

(ii) Observe that $q_{i0}^i \neq 0$ by (5) since $q_{i_t i}^0 = m_i > 0$ by (6). Now $\simeq$ is reflexive since $0 \in \mathscr{D}$ by Lemma 3.1(a). Note that $\simeq$ is symmetric by Lemma 3.1(a) since if $q_{ih}^j \neq 0$ for some $h \in \mathscr{D}$, then $q_{jh_t}^i \neq 0$ by (5). Finally if $i \simeq j$ and $j \simeq k$, then there are $h, h' \in \mathscr{D}$ such that $q_{ih}^j \neq 0$ and $q_{jh'}^k \neq 0$. This implies that $E_j(E_i \circ E_h) = |X|^{-1} q_{ih}^j E_j \neq 0$ and

$E_k(E_j \circ E_{h'}) = |X|^{-1} q^k_{jh'} E_k \neq 0$.Now $E_k(E_i \circ E_h \circ E_{h'}) = |X|^{-1} \sum_{\ell=0}^{d} q^{\ell}_{ih} E_k(E_\ell \circ E_{h'}) \neq 0$ since the summand $|X|^{-1} q^j_{ih} E_k(E_j \circ E_{h'}) \neq 0$. Since $0 \neq E_k(E_i \circ E_h \circ E_{h'}) = |X|^{-1} \sum_{\ell \in \mathscr{D}} q^{\ell}_{hh'} E_k(E_i \circ E_\ell)$, there is some $\ell \in \mathscr{D}$ such that $E_k(E_i \circ E_\ell) = |X|^{-1} q^k_{i\ell} E_k \neq 0$. Thus $i \simeq k$. $\square$

Lemma 4.2. *With reference to Lemma* 4.1(i), $|[\![X]\!]| = m_{\mathscr{D}}$ *and* $|[\![x]\!]| = k_{\mathscr{I}}$ *for all* $[\![x]\!] \in [\![X]\!]$.

Proof. Every row of $\sum_{i \in \mathscr{I}} A_i$ contains $\sum_{i \in \mathscr{I}} k_i = k_{\mathscr{I}}$ many ones, so $|[\![x]\!]| = k_{\mathscr{I}}$. Now there are a total of $|X|$ many points split evenly among blocks of size $k_{\mathscr{I}}$, so there are $|[\![X]\!]| = |X|/k_{\mathscr{I}} = m_{\mathscr{D}}$ many blocks by Lemma 3.4. $\square$

Lemma 4.3. *Suppose $\mathscr{M}$ is a $(d+1)$-dimensional Bose–Mesner algebra on X which is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$. Fix any $\sim$ equivalence class $[\![x]\!]$ of X. Then there is a Bose–Mesner algebra $[\![\mathscr{M}]\!]$ on $[\![x]\!]$ with Hadamard idempotents $\{[\![A_i]\!]\}_{i \in \mathscr{I}}$ and primitive idempotents $\{[\![E_i]\!] := [\![E]\!]_{[\![i]\!]}\}_{[\![i]\!] \in [\![\mathscr{D}]\!]}$ defined as follows.* (*In particular*, $|\mathscr{I}| = |[\![\mathscr{D}]\!]|$).

(i) *For each $i \in \mathscr{I}$, the Hadamard idempotent $[\![A_i]\!] \in \mathbb{M}_{[\![x]\!]}$ has entries $[\![A_i]\!](x', y') = A_i(x', y')$ $(x', y' \in [\![x]\!])$.*
(ii) *For each $[\![i]\!] \in [\![\mathscr{D}]\!]$, the primitive idempotent $[\![E_i]\!] \in \mathbb{M}_{[\![x]\!]}$ has entries $[\![E_i]\!](x', y') = \sum_{i' \in [\![i]\!]} E_{i'}(x', y')(x', y' \in [\![x]\!])$.*

$[\![\mathscr{M}]\!]$ *is called the* $[\![x]\!]$-block Bose–Mesner algebra of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$. *Moreover, the following hold.*

(a) *The intersection numbers of $[\![\mathscr{M}]\!]$ are $[\![p^h_{ij}]\!] = p^h_{ij}$ $(h, i, j \in \mathscr{I})$.*
(b) *The Krein parameters of $[\![\mathscr{M}]\!]$ are*

$$[\![q^h_{ij}]\!] := [\![q]\!]^{[\![h]\!]}_{[\![i]\!][\![j]\!]} = m_{\mathscr{D}}^{-1} \sum_{i' \in [\![i]\!], j' \in [\![j]\!]} q^{h'}_{i'j'} \quad ([\![h]\!], [\![i]\!], [\![j]\!] \in [\![\mathscr{D}]\!])$$

for any $h' \in [\![h]\!]$.

Proof. Observe that $[\![A_i]\!] \circ [\![A_j]\!] = \delta_{ij} [\![A_i]\!]$ $(i, j \in \mathscr{I})$ by (1) and construction. By Lemma 3.1, $[\![A_i]\!][\![A_j]\!] = \sum_{h \in \mathscr{I}} p^h_{ij} [\![A_h]\!]$ $(i, j \in \mathscr{I})$. Lemma 3.1 also implies that $\{[\![A_i]\!]\}_{i \in \mathscr{I}}$ contains the identity and is closed under transposition. Finally, by the definition of $\sim$, the $[\![A_i]\!]$ $(i \in \mathscr{I})$ sum to the all-ones matrix of $\mathbb{M}_{[\![x]\!]}$. Thus the span of $\{[\![A_i]\!]\}_{i \in \mathscr{I}}$ is a Bose–Mesner algebra $[\![\mathscr{M}]\!]$ and $\{[\![A_i]\!]\}_{i \in \mathscr{I}}$ is its set of Hadamard idempotents.

Now the Bose–Mesner algebra $[\![\mathscr{M}]\!]$ has a set of $|\mathscr{I}|$-many primitive idempotents, say $\{[\![F_i]\!]\}_{i=0}^{|\mathscr{I}|-1}$. To show that $\{[\![F_i]\!]\}_{i=0}^{|\mathscr{I}|-1} = \{[\![E_i]\!]\}_{[\![i]\!] \in [\![\mathscr{D}]\!]}$, we show that each $[\![E_i]\!]$ $([\![i]\!] \in [\![\mathscr{D}]\!])$ is a sum of some distinct $[\![F_i]\!]$ and vice versa.

Observe that by Lemma 3.1(c) and (3),

$$\sum_{i' \in [\![i]\!]} E_{i'} \circ \sum_{j \in \mathscr{I}} A_j = k_{\mathscr{I}} \sum_{i' \in [\![i]\!]} E_{i'} \circ \sum_{j \in \mathscr{D}} E_j = \frac{k_{\mathscr{I}}}{|X|} \sum_{i' \in [\![i]\!]} \sum_{j \in \mathscr{D}} \sum_{h=0}^{d} q^h_{i'j} E_h.$$

But $k_{\mathscr{I}}/|X| = m_{\mathscr{D}}^{-1}$ by Lemma 4.2 and the only nonzero $q^h_{i'j}$ are those with $h \in [\![i]\!]$ by Lemma 4.1(ii) since each $i' \in [\![i]\!]$ and $j \in \mathscr{D}$. Thus $(\sum_{i' \in [\![i]\!]} E_{i'}) \circ (\sum_{j \in \mathscr{I}} A_j) = m_{\mathscr{D}}^{-1} \sum_{h \in [\![i]\!]} (\sum_{j \in \mathscr{D}} \sum_{i' \in [\![i]\!]} q^h_{i'j}) E_h$. Now since $h \in [\![i]\!]$, Lemma 4.1(ii) and (7) give $\sum_{i' \in [\![i]\!]} q^h_{i'j} = \sum_{i'=0}^{d} q^h_{i'j} = m_j$. Thus $\sum_{j \in \mathscr{D}} \sum_{i' \in [\![i]\!]} q^h_{i'j} = m_{\mathscr{D}}$. Hence $\sum_{i' \in [\![i]\!]} E_{i'} \circ \sum_{j \in \mathscr{I}} A_j = \sum_{i' \in [\![i]\!]} E_{i'}$. Thus $\sum_{i' \in [\![i]\!]} E_{i'} \in \operatorname{span}\{A_j\}_{j \in \mathscr{I}}$, so $[\![E_i]\!] \in [\![\mathscr{M}]\!]$ by restricting to $[\![x]\!]$. Observe that $[\![E_i]\!][\![E_j]\!] = \delta_{[\![i]\!][\![j]\!]} [\![E_i]\!]$ as $(\sum_{i' \in [\![i]\!]} E_{i'})(\sum_{j' \in [\![j]\!]} E_{j'}) = \delta_{[\![i]\!][\![j]\!]} \sum_{i' \in [\![i]\!]} E_{i'}$. Hence each $[\![E_i]\!]$ $([\![i]\!] \in [\![\mathscr{D}]\!])$ is the sum of some distinct primitive idempotents of $[\![\mathscr{M}]\!]$.

We now show that each primitive idempotent is the sum of some of the $[\![E_i]\!]$. Since the nonzero entries of the A_i $(i \in \mathscr{I})$ are all indexed by elements of $[\![x]\!]$, each $[\![F_i]\!]$ has (u, v)-entry $[\![F_i]\!](u, v) = \sum_{i' \in [i]} E_{i'}(u, v)$ $(u, v \in [\![x]\!])$

for some subset $[i]$ of $\{0, 1, \ldots, d\}$. Compute as above

$$\sum_{i'\in[i]} E_{i'} \circ \sum_{j\in\mathscr{I}} A_j = k_{\mathscr{I}} \sum_{i'\in[i]} E_{i'} \circ \sum_{j\in\mathscr{D}} E_j = k_{\mathscr{I}}|X|^{-1} \sum_{i'\in[i]} \sum_{j\in\mathscr{D}} \sum_{h=0}^{d} q^h_{i'j} E_h$$
$$= m_{\mathscr{D}}^{-1} \sum_{i'\in[i]} \sum_{j\in\mathscr{D}} \sum_{h\in\llbracket i'\rrbracket} q^h_{i'j} E_h.$$

Restricting the left side of this equation to vertices indexed by $\llbracket x\rrbracket$ gives $\llbracket F_i\rrbracket \circ \llbracket J\rrbracket = \llbracket F_i\rrbracket$, so the right side must also restrict to give $\llbracket F_i\rrbracket$. Thus, in light of (5) and the definition of $\simeq$, the set $[i]$ must be a union of some of the classes in $\llbracket \mathscr{D}\rrbracket$. That is to say, each primitive idempotent is the sum of some of the $\llbracket E_i\rrbracket$, as required.

Observe that by Lemma 3.3, $\llbracket E_0\rrbracket = k_{\mathscr{I}}^{-1}\llbracket J\rrbracket$, where $\llbracket J\rrbracket$ denotes the all-ones matrix of $\mathbb{M}_{\llbracket x\rrbracket}$. Furthermore, $I = \sum_{h=0}^{d} E_h = \sum_{\llbracket h\rrbracket\in\llbracket\mathscr{D}\rrbracket} \sum_{h'\in\llbracket h\rrbracket} E_{h'}$, so restricting to points in $\llbracket x\rrbracket$ gives $\sum_{\llbracket h\rrbracket\in\llbracket\mathscr{D}\rrbracket} \llbracket E_h\rrbracket = \llbracket I\rrbracket$, the identity of $\mathbb{M}_{\llbracket x\rrbracket}$. Now the Krein parameters of $\llbracket \mathscr{M}\rrbracket$ are computed from the equation $\llbracket E_i\rrbracket \circ \llbracket E_j\rrbracket = k_{\mathscr{I}}^{-1} \sum \llbracket q^h_{ij}\rrbracket \llbracket E_h\rrbracket$ by first computing the product $\sum_{i'\in\llbracket i\rrbracket} E_{i'} \circ \sum_{j'\in\llbracket j\rrbracket} E_{j'}$ using (3) and then restricting to $\llbracket x\rrbracket$. □

The terminology "$\llbracket x\rrbracket$-block Bose–Mesner algebra of $\mathscr{M}$" is nonstandard. The literature generally uses "association subscheme" to refer to the association scheme related to a block Bose–Mesner algebra, but has no succinct means of referring to the Bose–Mesner algebra of a particular association subscheme. This distinction is generally not necessary as all block Bose–Mesner algebras with respect to a fixed set $\mathscr{I}$ of an imprimitive Bose–Mesner algebra are isomorphic as Bose–Mesner algebras, even though the related association schemes need not be isomorphic. We need this distinction as there is no reason to expect that the associated subconstituent algebras are isomorphic.

Lemma 4.4. *Suppose $\mathscr{M}$ is a $(d+1)$-dimensional Bose–Mesner algebra on X which is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$. Fix $p \in X$, and let $\llbracket \mathscr{M}\rrbracket$ denote the $\llbracket p\rrbracket$-block Bose–Mesner algebra $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$. Let $\llbracket \mathscr{M}^*\rrbracket$ denote the dual Bose–Mesner algebra of $\llbracket \mathscr{M}\rrbracket$ with respect to p.*

(i) *For each $i \in \mathscr{I}$, the dual idempotent $\llbracket E_i^*\rrbracket = \rho(\llbracket A_i\rrbracket) \in \mathbb{M}_{\llbracket p\rrbracket}$ of $\llbracket \mathscr{M}^*\rrbracket$ has entries $\llbracket E_i^*\rrbracket(x,x) = E_i^*(x,x)$ $(x \in \llbracket p\rrbracket)$.*
(ii) *For each $\llbracket i\rrbracket \in \llbracket \mathscr{D}\rrbracket$, the dual Hadamard idempotent $\llbracket A_i^*\rrbracket = \rho(k_{\mathscr{I}}\llbracket E_i\rrbracket) \in \mathbb{M}_{\llbracket p\rrbracket}$ of $\llbracket \mathscr{M}^*\rrbracket$ has entries $\llbracket A_i^*\rrbracket(x,x) = m_{\mathscr{D}}^{-1} \sum_{i'\in\llbracket i\rrbracket} A_{i'}^*(x,x)$ $(x \in \llbracket p\rrbracket)$.*

Proof. Straightforward from construction and Lemma 4.3. □

5. Quotient Bose–Mesner algebras

A quotient Bose–Mesner algebra can be constructed from an imprimitive Bose–Mesner algebra in a manner dual to that of the block Bose–Mesner algebras.

Lemma 5.1. *Suppose $\mathscr{M}$ is a $(d+1)$-dimensional Bose–Mesner algebra on X which is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$. Then the following are equivalence relations.*

(i) *The relation $\approx$ on X defined by $x \approx y$ if and only if $E_i(w,x) = E_i(w,y)$ for all $i \in \mathscr{D}$ and all $w \in X$. Let $\llbracket X\rrbracket$ denote the set of $\approx$ equivalence classes, and write $\llbracket x\rrbracket$ to denote the $\approx$ equivalence class of $x \in X$.*
(ii) *The relation $\cong$ on $\{0, 1, \ldots, d\}$ defined by $i \cong j$ if and only if $p^j_{ih} \neq 0$ for some $h \in \mathscr{I}$. Let $\llbracket \mathscr{I}\rrbracket$ denote the set of $\cong$ equivalence classes, and write $\llbracket i\rrbracket$ to denote the $\cong$ equivalence class of $i \in \{0, 1, \ldots, d\}$.*

Proof. (i) The relation $\approx$ is clearly reflexive, symmetric, and transitive. (ii) Use the proof of Lemma 4.1(ii) with the Kawada–Delsarte dual of each statement. □

Lemma 5.2. *With reference to Lemma* 5.1, *the equivalence relations $\sim$ of Lemma* 4.1(i) *and $\approx$ of Lemma* 5.1(i) *on X coincide.*

Proof. Observe that $(\sum_{i\in\mathscr{I}} A_i)(z,w)=1$ if $z\sim w$ and 0 otherwise. Thus by Lemma 3.3, $(\sum_{i\in\mathscr{D}} E_i)(z,w)=k_{\mathscr{I}}^{-1}$ if $z\sim w$ and 0 otherwise. Suppose $x\sim y$. Then columns x and y of $\sum_{i\in\mathscr{D}} E_i$ are the same. Let V denote the set of complex column vectors whose entries are indexed by X, and for all $z\in X$, let $\hat{z}\in V$ denote the vector with all entries 0 except for that indexed by z which is one. Then $\sum_{i\in\mathscr{D}}(E_i\hat{x})=(\sum_{i\in\mathscr{D}} E_i)\hat{x}=(\sum_{i\in\mathscr{D}} E_i)\hat{y}=\sum_{i\in\mathscr{D}}(E_i\hat{y})$. Since the E_i are projections onto orthogonal subspaces, it follows that $E_i\hat{x}=E_i\hat{y}$ for all $i\in\mathscr{D}$. Hence $x\approx y$. Now suppose $x\approx y$. Then $E_i\hat{x}=E_i\hat{y}$ for all $i\in\mathscr{D}$, so $(\sum_{i\in\mathscr{D}} E_i)\hat{x}=(\sum_{i\in\mathscr{D}} E_i)\hat{y}$. It follows that columns x and y of $\sum_{i\in\mathscr{I}} A_i$ are the same, so $x\sim y$. □

Lemma 5.3. *With reference to Lemma* 5.1(i), $|[\![X]\!]|=m_{\mathscr{D}}$ *and* $|[\![x]\!]|=k_{\mathscr{I}}$ *for all* $[\![x]\!]\in[\![X]\!]$.

Proof. Immediate from Lemmas 4.2 and 5.2. □

Lemma 5.4. *Suppose $\mathscr{M}$ is a $(d+1)$-dimensional Bose–Mesner algebra on X which is imprimitive with respect to $(\mathscr{I},\mathscr{D})$. Then there is a Bose–Mesner algebra $[\![\mathscr{M}]\!]$ on $[\![X]\!]$ with Hadamard idempotents $\{[\![A_i]\!]:=[\![A]\!]_{[\![i]\!]}\}_{[\![i]\!]\in[\![\mathscr{I}]\!]}$ and primitive idempotents $\{[\![E_i]\!]\}_{i\in\mathscr{D}}$ defined as follows. (In particular $|\mathscr{D}|=|[\![\mathscr{I}]\!]|$).*

(i) *For each $i\in\mathscr{D}$, the primitive idempotent $[\![E_i]\!]\in\mathbb{M}_{[\![X]\!]}$ has entries $[\![E_i]\!]([\![x]\!],[\![y]\!])=k_{\mathscr{I}}E_i(x',y')$ $([\![x]\!],[\![y]\!]\in[\![X]\!])$ for any $x'\in[\![x]\!]$, $y'\in[\![y]\!]$.*
(ii) *For each $[\![i]\!]\in[\![\mathscr{I}]\!]$, the Hadamard idempotent $[\![A_i]\!]\in\mathbb{M}_{[\![X]\!]}$ has entries $[\![A_i]\!]([\![x]\!],[\![y]\!])=\sum_{i'\in[\![i]\!]}A_{i'}(x',y')$ $([\![x]\!],[\![y]\!]\in[\![X]\!])$ for any $x'\in[\![x]\!]$, $y'\in[\![y]\!]$.*

$[\![\mathscr{M}]\!]$ *is called the* quotient Bose–Mesner algebra of $\mathscr{M}$ with respect to $(\mathscr{I},\mathscr{D})$. *Moreover, the following hold.*

(a) *The Krein parameters of $[\![\mathscr{M}]\!]$ are $[\![q_{ij}^h]\!]=q_{ij}^h$ $(h,i,j\in\mathscr{D})$.*
(b) *The intersection numbers of $[\![\mathscr{M}]\!]$ are*

$$[\![p_{ij}^h]\!]:=[\![p]\!]_{[\![i]\!][\![j]\!]}^{[\![h]\!]}=k_{\mathscr{I}}^{-1}\sum_{i'\in[\![i]\!],j'\in[\![j]\!]}p_{i'j'}^{h'}\quad([\![h]\!],[\![i]\!],[\![j]\!]\in[\![\mathscr{I}]\!])$$

for any $h'\in[\![h]\!]$.

Proof. Use the proof of Lemma 4.3 with the Kawada–Delsarte dual of each statement. □

Lemma 5.5. *Suppose $\mathscr{M}$ is a $(d+1)$-dimensional Bose–Mesner algebra on X which is imprimitive with respect to $(\mathscr{I},\mathscr{D})$. Let $[\![\mathscr{M}]\!]$ denote the quotient Bose–Mesner algebra of $\mathscr{M}$ with respect to $(\mathscr{I},\mathscr{D})$. Let $[\![\mathscr{M}^*]\!]$ denote the dual Bose–Mesner algebra of $[\![\mathscr{M}]\!]$ with respect to $[\![p]\!]$.*

(i) *For each $i\in\mathscr{D}$, the dual Hadamard idempotent $[\![A_i^*]\!]=\rho(m_{\mathscr{D}}[\![E_i]\!])\in\mathbb{M}_{[\![X]\!]}$ of $[\![\mathscr{M}^*]\!]$ has entries $[\![A_i^*]\!]([\![x]\!],[\![x]\!])=A_i^*(x,x)$ $([\![x]\!]\in[\![X]\!])$.*
(ii) *For each $[\![i]\!]\in[\![\mathscr{I}]\!]$, the dual idempotent $[\![E_i^*]\!]=\rho([\![A_i]\!])\in\mathbb{M}_{[\![X]\!]}$ of $[\![\mathscr{M}^*]\!]$ has entries $[\![E_i^*]\!]([\![x]\!],[\![x]\!])=\sum_{i'\in[\![i]\!]}E_{i'}^*(x',x')([\![x]\!]\in[\![X]\!])$ for any $x'\in[\![x]\!]$.*

Proof. Clear from construction and Lemma 5.4. □

Although we have described generators of the subconstituent algebras of block and quotient Bose–Mesner algebras associated with an imprimitive Bose–Mesner algebra (Lemmas 4.3, 4.4, 5.4, and 5.5), we have not described the structure of these subconstituent algebras. Since subconstituent algebras are semisimple, this is generally done by describing the isomorphism classes of the irreducible modules and the multiplicity with which they appear in the standard module. In a few cases the subconstituent algebras of block and quotient Bose–Mesner algebras have been related to that of the imprimitive Bose–Mesner algebra [5].

6. Duality of Bose–Mesner algebras

We recall some descriptions of formal and hyper-duality of Bose–Mesner algebras. See [2,3,20] for summaries of formal duality and [6,10] for more on hyper-duality. We work in the following setting.

Notation 6.1. Let X and $\widetilde{X}$ denote finite nonempty sets of the same size $|X| = |\widetilde{X}| = n$. Let $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ denote $(d+1)$-dimensional Bose–Mesner algebras on X and $\widetilde{X}$, respectively. Fix $p \in X$ and $\widetilde{p} \in \widetilde{X}$, and let $\mathscr{T}$ and $\widetilde{\mathscr{T}}$ denote the subconstituent algebras of $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ with respect to p and $\widetilde{p}$, respectively. Write $\widetilde{\ }$ with all objects associated with $\widetilde{\mathscr{T}}$, but otherwise use the same notation introduced for Bose–Mesner and their subconstituent algebras. Write τ and $\widetilde{\tau}$ to denote the transposition maps on $\mathbb{M}_X$ and $\mathbb{M}_{\widetilde{X}}$, respectively. Write $\mathbb{M}_{X,\widetilde{X}}$ to denote the set of complex matrices with rows indexed by X and columns indexed by $\widetilde{X}$.

Lemma 6.2 (*Bannai and Ito [2], Brouwer et al. [3], Neumaier [20]*). *With Notation* 6.1, *let* $\Psi : \mathscr{M} \to \widetilde{\mathscr{M}}$ *denote a linear bijection. Then the following are equivalent*:

(i) *For all* $A, B \in \mathscr{M}$,

$$\Psi(AB) = \Psi(A) \circ \Psi(B), \quad \Psi(A \circ B) = n^{-1}\Psi(A)\Psi(B). \tag{8}$$

(ii) *There exist orderings of the Hadamard and primitive idempotents of* $\mathscr{M}$ *and* $\widetilde{\mathscr{M}}$ *such that*

$$\Psi(E_i) = \widetilde{A}_i, \quad \Psi(A_i) = n\,{}^{\mathrm{t}}\widetilde{E}_i \quad (0 \leqslant i \leqslant d). \tag{9}$$

Suppose (i) *and* (ii) *hold. Then the map* Ψ *is called a* formal duality *from* $\mathscr{M}$ *to* $\widetilde{\mathscr{M}}$. $\mathscr{M}$ *and* $\widetilde{\mathscr{M}}$ *are said to be* formally dual *whenever there exists a formal duality from one to the other. Orderings of the Hadamard and primitive idempotents which satisfy* (9) *are said to be* standard *for* Ψ.

Lemma 6.3 (*Bannai and Ito [2], Brouwer et al. [3], Neumaier [20]*). *With Notation* 6.1, *the following are equivalent*:

(i) $\mathscr{M}$ *and* $\widetilde{\mathscr{M}}$ *are formally dual and their Hadamard and primitive idempotents are in standard orders for some formal duality from* $\mathscr{M}$ *to* $\widetilde{\mathscr{M}}$.
(ii) $p^h_{ij} = \widetilde{q}^h_{ij}$ *and* $q^h_{ij} = \widetilde{p}^h_{ij}$ $(0 \leqslant h, i, j \leqslant d)$.

Theorem 6.4 (*Curtin [6]*). *With Notation* 6.1, *suppose* $\psi : \mathscr{T} \to \widetilde{\mathscr{T}}$ *is an algebra isomorphism. Then the following are equivalent*:

(i) $\psi(\mathscr{M}) = \widetilde{\mathscr{M}}^*$, $\psi(\mathscr{M}^*) = \widetilde{\mathscr{M}}$, $\Psi := n\psi\rho|_{\mathscr{M}}$ *is a formal duality from* $\mathscr{M}$ *to* $\widetilde{\mathscr{M}}$, $\widetilde{\Psi} := n\tau\psi^{-1}\widetilde{\rho}|_{\widetilde{\mathscr{M}}}$ *is a formal duality from* $\widetilde{\mathscr{M}}$ *to* $\mathscr{M}$, $\Psi\widetilde{\Psi} = n\widetilde{\tau}|_{\widetilde{\mathscr{M}}}$, *and* $\widetilde{\Psi}\Psi = n\tau|_{\mathscr{M}}$.
(ii) *There exist orderings of the Hadamard and primitive idempotents of* $\mathscr{M}$ *and* $\widetilde{\mathscr{M}}$ *such that for all* i $(0 \leqslant i \leqslant d)$

$$\psi({}^{\mathrm{t}}A_i) = \widetilde{A}^*_i, \quad \psi(A^*_i) = \widetilde{A}_i, \quad \psi(E_i) = \widetilde{E}^*_i, \quad \psi(E^*_i) = {}^{\mathrm{t}}\widetilde{E}_i. \tag{10}$$

Suppose (i) *and* (ii) *hold. Then the map* ψ *is called a* hyper-duality *from* $\mathscr{T}$ *to* $\widetilde{\mathscr{T}}$. $\mathscr{T}$ *and* $\widetilde{\mathscr{T}}$ *are said to be* hyper-dual *to one another whenever there exists a hyper-duality from one to the other. Orderings of the Hadamard and primitive idempotents which satisfy* (10) *are said to be* standard *for* ψ. *Standard orderings for* ψ *are standard for the formal dualities* Ψ *and* $\widetilde{\Psi}$ *of* (i).

In the presence of formal duality, it is not necessary to check all four conditions in Eq. (10).

Lemma 6.5 (*Curtin [6]*). *With Notation* 6.1, *assume that* $\mathscr{M}$ *and* $\widetilde{\mathscr{M}}$ *are formally dual to one another and that the Hadamard and primitive idempotents are in standard orders for some formal duality of* $\mathscr{M}$ *and* $\widetilde{\mathscr{M}}$.

(i) *Suppose* $\psi : \mathscr{M}^* \to \widetilde{\mathscr{M}}$ *is a linear map. Then* $\psi(E^*_i) = {}^{\mathrm{t}}\widetilde{E}_i$ $(0 \leqslant i \leqslant d)$ *if and only if* $\psi(A^*_i) = \widetilde{A}_i$ $(0 \leqslant i \leqslant d)$.
(ii) *Suppose* $\psi : \mathscr{M} \to \widetilde{\mathscr{M}}^*$ *is a linear map. Then* $\psi({}^{\mathrm{t}}A_i) = \widetilde{A}^*_i$ $(0 \leqslant i \leqslant d)$ *if and only if* $\psi(E_i) = \widetilde{E}^*_i$ $(0 \leqslant i \leqslant d)$.

Definition 6.6. With Notation 6.1, suppose that $\psi : \mathscr{T} \to \widetilde{\mathscr{T}}$ is a hyper-duality. We say that an invertible matrix $H \in \mathbb{M}_{X,\widetilde{X}}$ *represents* the hyper-duality ψ whenever $\psi(A) = H^{-1}AH$ for all $A \in \mathscr{T}$. In this case we say that ψ is *representable*, and that $\mathscr{T}$ and $\widetilde{\mathscr{T}}$ are *representably hyper-dual* to one another.

7. Duality and imprimitive Bose–Mesner algebras

We describe the inheritance of formal and hyper-duality by block and quotient Bose–Mesner algebras from an imprimitive Bose–Mesner algebra. We consider a Bose–Mesner algebra $\mathscr{M}$ which is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$ and block and quotient Bose–Mesner algebras $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor$ and $\lceil\!\lceil \mathscr{M} \rceil\!\rceil$ of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$. We also consider the formal dual $\widetilde{\mathscr{M}}$ of $\mathscr{M}$ which turns out to be imprimitive with respect to $(\mathscr{D}, \mathscr{I})$, so there are block and quotient Bose–Mesner algebras $\widetilde{\lfloor\!\lfloor \widetilde{\mathscr{M}} \rfloor\!\rfloor}$ and $\widetilde{\lceil\!\lceil \widetilde{\mathscr{M}} \rceil\!\rceil}$ of $\widetilde{\mathscr{M}}$ with respect to $(\mathscr{D}, \mathscr{I})$. The notation aside, the inheritance of formal duality is straightforward and well known (cf. [3]).

Theorem 7.1. *With Notation* 6.1, *assume that $\Psi : \mathscr{M} \to \widetilde{\mathscr{M}}$ is a formal duality. Suppose $\mathscr{M}$ is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$. Then the following hold.*

(i) *$\widetilde{\mathscr{M}}$ is imprimitive with respect to $(\mathscr{D}, \mathscr{I})$.*
(ii) *Let $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor$ denote any block Bose–Mesner algebra of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$, and let $\widetilde{\lceil\!\lceil \widetilde{\mathscr{M}} \rceil\!\rceil}$ denote the quotient Bose–Mesner algebra of $\widetilde{\mathscr{M}}$ with respect to $(\mathscr{D}, \mathscr{I})$. Then the linear map $\lfloor\!\lfloor \Psi \widetilde{\rceil\!\rceil} : \lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor \to \widetilde{\lceil\!\lceil \widetilde{\mathscr{M}} \rceil\!\rceil}$ satisfying $\lfloor\!\lfloor \Psi \widetilde{\rceil\!\rceil}(\lfloor\!\lfloor A_i \rfloor\!\rfloor) = n^{\mathrm{t}}\widetilde{\lceil\!\lceil \widetilde{E}_i \rceil\!\rceil}$ $(i \in \mathscr{I})$ is a formal duality, where $\{A_i\}_{i\in\mathscr{I}}$ and $\{\widetilde{E}_i\}_{i\in\mathscr{I}}$ are in standard orders for Ψ.*
(iii) *Let $\lceil\!\lceil \mathscr{M} \rceil\!\rceil$ denote the quotient Bose–Mesner algebra of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$, and let $\widetilde{\lfloor\!\lfloor \widetilde{\mathscr{M}} \rfloor\!\rfloor}$ denote any block Bose–Mesner algebra of $\widetilde{\mathscr{M}}$ with respect to $(\mathscr{D}, \mathscr{I})$. Then the linear map $\lceil\!\lceil \Psi \widetilde{\rfloor\!\rfloor} : \lceil\!\lceil \mathscr{M} \rceil\!\rceil \to \widetilde{\lfloor\!\lfloor \widetilde{\mathscr{M}} \rfloor\!\rfloor}$ satisfying $\lceil\!\lceil \Psi \widetilde{\rfloor\!\rfloor}(\lceil\!\lceil E_i \rceil\!\rceil) = \widetilde{\lfloor\!\lfloor \widetilde{A}_i \rfloor\!\rfloor}$ $(i \in \mathscr{D})$ is a formal duality, where $\{E_i\}_{i\in\mathscr{D}}$ and $\{\widetilde{A}_i\}_{i\in\mathscr{D}}$ are in standard orders for Ψ.*

Proof. Assume that the Hadamard and primitive idempotents are in standard orders for Ψ. Then $p^h_{ij} = \widetilde{q}^h_{ij}$ and $q^h_{ij} = \widetilde{p}^h_{ij}$ by Lemma 6.3. In particular, $p^h_{ij} = 0$ if and only if $\widetilde{q}^h_{ij} = 0$. Thus $\widetilde{\mathscr{M}}$ is imprimitive with respect to $(\mathscr{D}, \mathscr{I})$. Moreover, $\lfloor\!\lfloor p^h_{ij} \rfloor\!\rfloor = \widetilde{\lceil\!\lceil \widetilde{q}^h_{ij} \rceil\!\rceil}$ and $\lfloor\!\lfloor q^h_{ij} \rfloor\!\rfloor = \widetilde{\lceil\!\lceil \widetilde{p}^h_{ij} \rceil\!\rceil}$ by Lemmas 4.3 and 5.4. Thus (ii) and (iii) hold by Lemma 6.3. $\square$

The formally dual Bose–Mesner algebras $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor$ and $\widetilde{\lceil\!\lceil \widetilde{\mathscr{M}} \rceil\!\rceil}$ have respective point sets $\lfloor\!\lfloor x \rfloor\!\rfloor$ and $\widetilde{\lceil\!\lceil \widetilde{X} \rceil\!\rceil}$, both of size $k_{\mathscr{I}} = \widetilde{m}_{\mathscr{I}}$. Similarly, the formally dual Bose–Mesner algebras $\lceil\!\lceil \mathscr{M} \rceil\!\rceil$ and $\widetilde{\lfloor\!\lfloor \widetilde{\mathscr{M}} \rfloor\!\rfloor}$ have respective point sets $\lceil\!\lceil X \rceil\!\rceil$ and $\widetilde{\lfloor\!\lfloor \widetilde{x} \rfloor\!\rfloor}$, both of size $m_{\mathscr{D}} = \widetilde{k}_{\mathscr{D}}$.

Theorem 7.2. *With Notation* 6.1, *suppose that $\psi : \mathscr{T} \to \widetilde{\mathscr{T}}$ is a hyper-duality. Further suppose that $\mathscr{M}$ is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$ (so $\widetilde{\mathscr{M}}$ is imprimitive with respect to $(\mathscr{D}, \mathscr{I})$).*

(i) *Let $\lfloor\!\lfloor \mathscr{T} \rfloor\!\rfloor$ denote the subconstituent algebra with respect to p of the $\lfloor\!\lfloor p \rfloor\!\rfloor$-block Bose–Mesner algebra $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor$ of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$. Let $\widetilde{\lceil\!\lceil \widetilde{\mathscr{T}} \rceil\!\rceil}$ denote the subconstituent algebra with respect to $\widetilde{\lceil\!\lceil \widetilde{p} \rceil\!\rceil}$ of the quotient Bose–Mesner algebra $\widetilde{\lceil\!\lceil \widetilde{\mathscr{M}} \rceil\!\rceil}$ of $\widetilde{\mathscr{M}}$ with respect to $(\mathscr{D}, \mathscr{I})$. Then there is a hyper-duality $\lfloor\!\lfloor \psi \widetilde{\rceil\!\rceil} : \lfloor\!\lfloor \mathscr{T} \rfloor\!\rfloor \to \widetilde{\lceil\!\lceil \widetilde{\mathscr{T}} \rceil\!\rceil}$ satisfying $\lfloor\!\lfloor \psi \widetilde{\rceil\!\rceil}({}^{\mathrm{t}}\lfloor\!\lfloor A_i \rfloor\!\rfloor) = \widetilde{\lceil\!\lceil \widetilde{A}^*_i \rceil\!\rceil}$, $\lfloor\!\lfloor \psi \widetilde{\rceil\!\rceil}(\lfloor\!\lfloor E^*_i \rfloor\!\rfloor) = {}^{\mathrm{t}}\widetilde{\lceil\!\lceil \widetilde{E}_i \rceil\!\rceil}$ $(i \in \mathscr{I})$, where $\{A_i\}_{i\in\mathscr{I}}$ and $\{\widetilde{E}_i\}_{i\in\mathscr{I}}$ are in standard orders for ψ.*
(ii) *Let $\lceil\!\lceil \mathscr{T} \rceil\!\rceil$ denote the subconstituent algebra with respect to $\lceil\!\lceil p \rceil\!\rceil$ of the quotient Bose–Mesner algebra $\lceil\!\lceil \mathscr{M} \rceil\!\rceil$ of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$. Let $\widetilde{\lfloor\!\lfloor \widetilde{\mathscr{T}} \rfloor\!\rfloor}$ denote the subconstituent algebra with respect to $\widetilde{p}$ of the $\widetilde{\lfloor\!\lfloor \widetilde{p} \rfloor\!\rfloor}$-block Bose–Mesner algebra $\widetilde{\lfloor\!\lfloor \widetilde{\mathscr{M}} \rfloor\!\rfloor}$ of $\widetilde{\mathscr{M}}$ with respect to $(\mathscr{D}, \mathscr{I})$. Then there is a hyper-duality $\lceil\!\lceil \psi \widetilde{\rfloor\!\rfloor} : \lceil\!\lceil \mathscr{T} \rceil\!\rceil \to \widetilde{\lfloor\!\lfloor \widetilde{\mathscr{T}} \rfloor\!\rfloor}$ satisfying $\lceil\!\lceil \psi \widetilde{\rfloor\!\rfloor}(\lceil\!\lceil E_i \rceil\!\rceil) = \widetilde{\lfloor\!\lfloor \widetilde{E}^*_i \rfloor\!\rfloor}$, $\lceil\!\lceil \psi \widetilde{\rfloor\!\rfloor}(\lceil\!\lceil A^*_i \rceil\!\rceil) = \widetilde{\lfloor\!\lfloor \widetilde{A}_i \rfloor\!\rfloor}$ $(i \in \mathscr{D})$, where $\{E_i\}_{i\in\mathscr{D}}$ and $\{\widetilde{A}_i\}_{i\in\mathscr{D}}$ are in standard orders for ψ.*

Proof. (i) Assume that the Hadamard and primitive idempotents of $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ are in standard orders for the hyper-duality ψ. Let $\mathscr{T}_{\mathscr{I}}$ denote the subalgebra of $\mathscr{T}$ generated by $\{A_i\}_{i\in\mathscr{I}} \cup \{E^*_i\}_{i\in\mathscr{I}}$, and let $\widetilde{\mathscr{T}}_{\mathscr{I}}$ denote the subalgebra of $\widetilde{\mathscr{T}}$ generated by $\{\widetilde{E}_i\}_{i\in\mathscr{I}} \cup \{\widetilde{A}^*_i\}_{i\in\mathscr{I}}$. The nonzero entries of all of the generators of $\mathscr{T}_{\mathscr{I}}$ are indexed by elements of $\lfloor\!\lfloor p \rfloor\!\rfloor$ by Lemmas 3.1, 4.3, and 4.4. Hence the map $\lambda : \lfloor\!\lfloor \mathscr{T} \rfloor\!\rfloor \to \mathscr{T}_{\mathscr{I}}$ satisfying $\lambda({}^{\mathrm{t}}\lfloor\!\lfloor A_i \rfloor\!\rfloor) = {}^{\mathrm{t}}A_i$ and $\lambda(\lfloor\!\lfloor E^*_i \rfloor\!\rfloor) = E^*_i$ $(i \in \mathscr{I})$ is an isomorphism. Now $\psi({}^{\mathrm{t}}A_i) = \widetilde{A}^*_i$ and $\psi(E^*_i) = {}^{\mathrm{t}}\widetilde{E}_i$ $(i \in \mathscr{I})$, so ψ maps $\mathscr{T}_{\mathscr{I}}$ onto $\widetilde{\mathscr{T}}_{\mathscr{I}}$. Observe that by

Lemmas 5.4 and 5.5, the map $\mu : \widetilde{\mathscr{T}}_{\mathscr{I}} \to \widetilde{\lceil\!\lceil \widetilde{\mathscr{T}} \rceil\!\rceil}$ which restricts the matrices to a principal minor whose rows and columns are indexed by class representatives for $\lceil\!\lceil \widetilde{X} \rceil\!\rceil$ and multiplies by $\widetilde{k}_{\mathscr{D}}$ is an algebra isomorphism from $\widetilde{\mathscr{T}}_{\mathscr{I}}$ onto $\widetilde{\lceil\!\lceil \widetilde{\mathscr{T}} \rceil\!\rceil}$ satisfying $\mu(\widetilde{A}^*_i) = \widetilde{\lceil\!\lceil \widetilde{A}^*_i \rceil\!\rceil}$ and $\mu({}^{\mathrm{t}}\widetilde{E}_i) = {}^{\mathrm{t}}\widetilde{\lceil\!\lceil \widetilde{E}_i \rceil\!\rceil}$. Now set $\widetilde{\lfloor\!\lfloor \psi \rceil\!\rceil} = \mu\psi\lambda$, composing right to left. Then $\widetilde{\lfloor\!\lfloor \psi \rceil\!\rceil}$ is an algebra isomorphism from $\lfloor\!\lfloor \mathscr{T} \rfloor\!\rfloor$ onto $\widetilde{\lceil\!\lceil \widetilde{\mathscr{T}} \rceil\!\rceil}$ by construction.

By Theorems 6.4 and 7.1, $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor$ and $\widetilde{\lceil\!\lceil \widetilde{\mathscr{M}} \rceil\!\rceil}$ are formally dual and the orderings of the Hadamard and primitive idempotents induced by $\widetilde{\lfloor\!\lfloor \psi \rceil\!\rceil}({}^{\mathrm{t}}\lfloor\!\lfloor A_i \rfloor\!\rfloor) = \widetilde{\lceil\!\lceil \widetilde{A}^*_i \rceil\!\rceil}$ and $\widetilde{\lfloor\!\lfloor \psi \rceil\!\rceil}(\lfloor\!\lfloor E^*_i \rfloor\!\rfloor) = {}^{\mathrm{t}}\widetilde{\lceil\!\lceil \widetilde{E}_i \rceil\!\rceil}$ $(i \in \mathscr{I})$ are part of a standard ordering for some formal duality. Thus Lemma 6.5 implies that $\widetilde{\lfloor\!\lfloor \psi \rceil\!\rceil}(\lfloor\!\lfloor E_i \rfloor\!\rfloor) = \widetilde{\lceil\!\lceil \widetilde{E}^*_i \rceil\!\rceil}$ and $\widetilde{\lfloor\!\lfloor \psi \rceil\!\rceil}(\lfloor\!\lfloor A^*_i \rfloor\!\rfloor) = \widetilde{\lceil\!\lceil \widetilde{A}_i \rceil\!\rceil}$ for some orderings of $\{\lfloor\!\lfloor E_i \rfloor\!\rfloor\}_{\lfloor\!\lfloor i \rfloor\!\rfloor \in \lfloor\!\lfloor \mathscr{D} \rfloor\!\rfloor}$ and $\{\widetilde{\lceil\!\lceil \widetilde{A}_i \rceil\!\rceil}\}_{\widetilde{\lceil\!\lceil \widetilde{i} \rceil\!\rceil} \in \widetilde{\lceil\!\lceil \widetilde{\mathscr{D}} \rceil\!\rceil}}$.

(ii) Similar to (i). First invert the restriction and multiplication by $k_{\mathscr{I}}$ which gave $\lceil\!\lceil \mathscr{T} \rceil\!\rceil$ from $\mathscr{T}$. Then apply ψ, and finally restrict $\widetilde{\mathscr{T}}$ to $\widetilde{\lfloor\!\lfloor \widetilde{\mathscr{T}} \rfloor\!\rfloor}$. $\square$

Theorem 7.3. *With reference to Theorem* 7.2, *suppose that the matrix* $H \in \mathbb{M}_{X,\widetilde{X}}$ *represents the hyper-duality* $\psi : \mathscr{T} \to \widetilde{\mathscr{T}}$.

(i) *Let* $\widetilde{\lfloor\!\lfloor H \rceil\!\rceil} \in \mathbb{M}_{\lfloor\!\lfloor p \rfloor\!\rfloor, \widetilde{\lceil\!\lceil \widetilde{X} \rceil\!\rceil}}$ *have* $(x, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil})$*-entry*

$$\widetilde{\lfloor\!\lfloor H \rceil\!\rceil}(x, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}) = \sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} H(x, \widetilde{x}') \quad (x \in \lfloor\!\lfloor p \rfloor\!\rfloor, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil} \in \widetilde{\lceil\!\lceil \widetilde{X} \rceil\!\rceil}).$$

Then $\widetilde{\lfloor\!\lfloor H \rceil\!\rceil}$ *represents the hyper-duality* $\widetilde{\lfloor\!\lfloor \psi \rceil\!\rceil}$ *of Theorem* 7.2(i).

(ii) *Let* $\widetilde{\lceil\!\lceil H \rfloor\!\rfloor} \in \mathbb{M}_{\lceil\!\lceil X \rceil\!\rceil, \widetilde{\lfloor\!\lfloor \widetilde{p} \rfloor\!\rfloor}}$ *have* $(\lceil\!\lceil x \rceil\!\rceil, \widetilde{x})$*-entry*

$$\widetilde{\lceil\!\lceil H \rfloor\!\rfloor}(\lceil\!\lceil x \rceil\!\rceil, \widetilde{x}) = \sum_{x' \in \lceil\!\lceil x \rceil\!\rceil} H(x', \widetilde{x}) \quad (\lceil\!\lceil x \rceil\!\rceil \in \lceil\!\lceil X \rceil\!\rceil, \widetilde{x} \in \widetilde{\lfloor\!\lfloor \widetilde{p} \rfloor\!\rfloor}).$$

Then $\widetilde{\lceil\!\lceil H \rfloor\!\rfloor}$ *represents the hyper-duality* $\widetilde{\lceil\!\lceil \psi \rfloor\!\rfloor}$ *of Theorem* 7.2(i).

Proof. (i) Assume that the Hadamard and primitive idempotents of $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ are in standard orders for ψ. Fix $i \in \mathscr{I}$, $x \in \lfloor\!\lfloor p \rfloor\!\rfloor$, and $\widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil} \in \widetilde{\lceil\!\lceil \widetilde{X} \rceil\!\rceil}$. By Theorem 6.4, ${}^{\mathrm{t}}A_i H = H\widetilde{A}^*_i$, so $\sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} {}^{\mathrm{t}}A_i H(x, \widetilde{x}') = \sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} H\widetilde{A}^*_i(x, \widetilde{x}')$. Compute both sides of this equation. On the left side, for all $\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}$

$${}^{\mathrm{t}}A_i H(x, \widetilde{x}') = \sum_{w \in X} {}^{\mathrm{t}}A_i(x, w) H(w, \widetilde{x}') = \sum_{\lfloor\!\lfloor w \rfloor\!\rfloor \in \lfloor\!\lfloor X \rfloor\!\rfloor} \sum_{w' \in \lfloor\!\lfloor w \rfloor\!\rfloor} {}^{\mathrm{t}}A_i(x, w') H(w', \widetilde{x}').$$

But ${}^{\mathrm{t}}A_i(x, w') = 0$ unless $x \sim w'$ (i.e. $\lfloor\!\lfloor w \rfloor\!\rfloor = \lfloor\!\lfloor x \rfloor\!\rfloor = \lfloor\!\lfloor p \rfloor\!\rfloor$) since $i \in \mathscr{I}$. Thus ${}^{\mathrm{t}}A_i H(x, \widetilde{x}') = \sum_{w' \in \lfloor\!\lfloor p \rfloor\!\rfloor} {}^{\mathrm{t}}A_i(x, w') H(w', \widetilde{x}')$. Now ${}^{\mathrm{t}}A_i(x, w') = {}^{\mathrm{t}}\lfloor\!\lfloor A_i \rfloor\!\rfloor(x, w')$ by Lemma 4.3 as $x, w' \in \lfloor\!\lfloor p \rfloor\!\rfloor$, and $\sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} H(w', \widetilde{x}') = \widetilde{\lfloor\!\lfloor H \rceil\!\rceil}(w', \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil})$. Thus $\sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} {}^{\mathrm{t}}A_i H(x, \widetilde{x}') = {}^{\mathrm{t}}\lfloor\!\lfloor A_i \rfloor\!\rfloor \widetilde{\lfloor\!\lfloor H \rceil\!\rceil}(x, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil})$. On the right side, $H\widetilde{A}^*_i(x, \widetilde{x}') = H(x, \widetilde{x}')\widetilde{A}^*_i(\widetilde{x}', \widetilde{x}')$ for all $\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}$. Note that $\widetilde{A}^*_i(\widetilde{x}', \widetilde{x}') = \widetilde{\lceil\!\lceil \widetilde{A}^*_i \rceil\!\rceil}(\widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil})$ by Lemma 5.5. Thus $\sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} H\widetilde{A}^*_i(x, \widetilde{x}') = \widetilde{\lfloor\!\lfloor H \rceil\!\rceil}\widetilde{\lceil\!\lceil \widetilde{A}^*_i \rceil\!\rceil}(x, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil})$. Hence ${}^{\mathrm{t}}\lfloor\!\lfloor A_i \rfloor\!\rfloor \widetilde{\lfloor\!\lfloor H \rceil\!\rceil} = \widetilde{\lfloor\!\lfloor H \rceil\!\rceil}\widetilde{\lceil\!\lceil \widetilde{A}^*_i \rceil\!\rceil}$.

By Theorem 6.4, $E^*_i H = H{}^{\mathrm{t}}\widetilde{E}_i$, so $\sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} E^*_i H(x, \widetilde{x}') = \sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} H{}^{\mathrm{t}}\widetilde{E}_i(x, \widetilde{x}')$. Compute both sides of this equation. On the right side, for all $\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}$

$$H{}^{\mathrm{t}}\widetilde{E}_i(x, \widetilde{x}') = \sum_{\widetilde{w} \in \widetilde{X}} H(x, \widetilde{w}){}^{\mathrm{t}}\widetilde{E}_i(\widetilde{w}, \widetilde{x}') = \sum_{\widetilde{\lceil\!\lceil \widetilde{w} \rceil\!\rceil} \in \widetilde{\lceil\!\lceil \widetilde{X} \rceil\!\rceil}} \sum_{\widetilde{w}' \in \widetilde{\lceil\!\lceil \widetilde{w} \rceil\!\rceil}} H(x, \widetilde{w}'){}^{\mathrm{t}}\widetilde{E}_i(\widetilde{w}', \widetilde{x}').$$

By Lemma 5.4, ${}^{\mathrm{t}}\widetilde{E}_i(\widetilde{w}', \widetilde{x}') = \widetilde{k}_{\mathscr{D}}^{-1}\,{}^{\mathrm{t}}\widetilde{\lceil\!\lceil \widetilde{E}_i \rceil\!\rceil}(\widetilde{w}', \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil})$, independent of $\widetilde{x}'$ and $\widetilde{w}'$, so this term factors out of the sum over $\widetilde{w}' \in \widetilde{\lceil\!\lceil \widetilde{w} \rceil\!\rceil}$. Thus

$$\begin{aligned} H{}^{\mathrm{t}}\widetilde{E}_i(x, \widetilde{x}') &= \sum_{\widetilde{\lceil\!\lceil \widetilde{w} \rceil\!\rceil} \in \widetilde{\lceil\!\lceil \widetilde{X} \rceil\!\rceil}} \widetilde{k}_{\mathscr{D}}^{-1} \widetilde{\lfloor\!\lfloor H \rceil\!\rceil}(x, \widetilde{\lceil\!\lceil \widetilde{w} \rceil\!\rceil}){}^{\mathrm{t}}\widetilde{\lceil\!\lceil \widetilde{E}_i \rceil\!\rceil}(\widetilde{\lceil\!\lceil \widetilde{w} \rceil\!\rceil}, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}) \\ &= \widetilde{k}_{\mathscr{D}}^{-1} \widetilde{\lfloor\!\lfloor H \rceil\!\rceil}\,{}^{\mathrm{t}}\,\widetilde{\lceil\!\lceil \widetilde{E}_i \rceil\!\rceil}(x, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}). \end{aligned}$$

Note that this is independent of $\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}$, and that $|\widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}| = \widetilde{k}_{\mathscr{D}}$ by Lemma 5.3. Thus $\sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} H^{\mathrm{t}}\widetilde{E}_i(x, \widetilde{x}') = \widetilde{k}_{\mathscr{D}}\widetilde{k}_{\mathscr{D}}^{-1} \lfloor\!\lfloor H \widetilde{\rceil\!\rceil}{}^{\mathrm{t}}\widetilde{\lceil\!\lceil} \widetilde{E}_i \widetilde{\rceil\!\rceil}(x, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}) = \lfloor\!\lfloor H \widetilde{\rceil\!\rceil}{}^{\mathrm{t}}\widetilde{\lceil\!\lceil} \widetilde{E}_i \widetilde{\rceil\!\rceil}(x, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil})$.On the left side compute $\sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} E_i^* H(x, \widetilde{x}') = \sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} E_i^*(x, x)$ $H(x, \widetilde{x}') = E_i^*(x, x) \sum_{\widetilde{x}' \in \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}} H(x, \widetilde{x}') = \lfloor\!\lfloor E_i^* \rfloor\!\rfloor(x, x) \lfloor\!\lfloor H \widetilde{\rceil\!\rceil}(x, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil}) = \lfloor\!\lfloor E_i^* \rfloor\!\rfloor \lfloor\!\lfloor H \widetilde{\rceil\!\rceil}(x, \widetilde{\lceil\!\lceil \widetilde{x} \rceil\!\rceil})$. Hence $\lfloor\!\lfloor E_i^* \rfloor\!\rfloor \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} = \lfloor\!\lfloor H \widetilde{\rceil\!\rceil}{}^{\mathrm{t}}\widetilde{\lceil\!\lceil} \widetilde{E}_i \widetilde{\rceil\!\rceil}$.

By Theorems 6.4 and 7.1, $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor$ and $\widetilde{\lceil\!\lceil} \widetilde{\mathscr{M}} \widetilde{\rceil\!\rceil}$ are formally dual and the orderings of the Hadamard and primitive idempotents induced by ${}^{\mathrm{t}}\lfloor\!\lfloor A_i \rfloor\!\rfloor \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} = \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} \widetilde{\lceil\!\lceil} \widetilde{A}_i^* \widetilde{\rceil\!\rceil}$ and $\lfloor\!\lfloor E_i^* \rfloor\!\rfloor \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} = \lfloor\!\lfloor H \widetilde{\rceil\!\rceil}{}^{\mathrm{t}}\widetilde{\lceil\!\lceil} \widetilde{E}_i \widetilde{\rceil\!\rceil}$ for all $i \in \mathscr{I}$ are part of a standard ordering for some formal duality. Thus Lemma 6.5 implies that $\lfloor\!\lfloor E_i \rfloor\!\rfloor \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} = \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} \widetilde{\lceil\!\lceil} \widetilde{E}_i^* \widetilde{\rceil\!\rceil}$ and $\lfloor\!\lfloor A_i^* \rfloor\!\rfloor \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} = \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} \widetilde{\lceil\!\lceil} \widetilde{A}_i \widetilde{\rceil\!\rceil}$ for some orderings of $\{\lfloor\!\lfloor E_i \rfloor\!\rfloor\}_{\lfloor\!\lfloor i \rfloor\!\rfloor \in \lfloor\!\lfloor \mathscr{D} \rfloor\!\rfloor}$ and $\{\widetilde{\lceil\!\lceil} \widetilde{A}_i \widetilde{\rceil\!\rceil}\}_{\widetilde{\lceil\!\lceil} i \widetilde{\rceil\!\rceil} \in \widetilde{\lceil\!\lceil} \mathscr{D} \widetilde{\rceil\!\rceil}}$.

Note that H is invertible, and write L_H to denote the (invertible) linear transformation defined by left multiplication by H from the column vector space with entries indexed by $\widetilde{X}$ to the column vector space with entries indexed by X. Since $H^{\mathrm{t}}\widetilde{E}_i = E_i^* H$ $(i \in \mathscr{I})$, L_H maps the span of the column spaces of all $\widetilde{E}_i$ $(i \in \mathscr{I})$ onto that of all E_i^* $(i \in \mathscr{I})$. The column spaces of all $\widetilde{E}_i$ $(i \in \mathscr{I})$ are contained in the span of all column vectors with entries indexed by $\widetilde{X}$ which are constant on each class of $\widetilde{\lceil\!\lceil} \widetilde{X} \widetilde{\rceil\!\rceil}$ by Lemma 5.1(i). Equality holds since each of these vector spaces has dimension $\widetilde{m}_{\mathscr{I}}$. The span of the column spaces of all E_i^* $(i \in \mathscr{I})$ are precisely those column vectors with entries indexed by X which is zero in all entries not indexed by elements of $\lfloor\!\lfloor p \rfloor\!\rfloor$.

Consider the invertible linear transformation θ from the column vector space with entries indexed by $\widetilde{\lceil\!\lceil} \widetilde{X} \widetilde{\rceil\!\rceil}$ to the column vectors space with entries indexed by $\widetilde{X}$ which are constant on each class of $\widetilde{\lceil\!\lceil} \widetilde{X} \widetilde{\rceil\!\rceil}$ defined by $\theta(\widetilde{\vec{v}})(\widetilde{x}) = \widetilde{\lceil\!\lceil} \widetilde{\vec{v}} \widetilde{\rceil\!\rceil}(\widetilde{\lceil\!\lceil} \widetilde{x} \widetilde{\rceil\!\rceil})$ for all $\widetilde{x} \in \widetilde{X}$. Also consider the invertible linear transformation η from the column vector space with entries indexed by X with all entries not indexed by elements of $\lfloor\!\lfloor p \rfloor\!\rfloor$ equal to zero to the column vector with entries indexed by $\lfloor\!\lfloor p \rfloor\!\rfloor$ defined by $\eta(\vec{v})(x) = \lfloor\!\lfloor \vec{v} \rfloor\!\rfloor(x)$ for $x \in \lfloor\!\lfloor p \rfloor\!\rfloor$. Observe that the result of multiplication by $\lfloor\!\lfloor H \widetilde{\rceil\!\rceil}$ on the column vector space with entries indexed by $\widetilde{\lceil\!\lceil} \widetilde{X} \widetilde{\rceil\!\rceil}$ is just the composition $\eta L_H \theta$ of invertible linear transformations. Thus $\lfloor\!\lfloor H \widetilde{\rceil\!\rceil}$ is invertible.

(ii) Similar to (i). By Theorem 6.4, $E_i H = H\widetilde{E}_i^*$, so $\sum_{x' \in \lceil\!\lceil x \rceil\!\rceil} E_i H(x', \widetilde{x}) = \sum_{x' \in \lceil\!\lceil x \rceil\!\rceil} H\widetilde{E}_i^*(x', \widetilde{x})$. Computing both sides of this equation with simplifications similar to those used in (i) gives $\lceil\!\lceil E_i \rceil\!\rceil \lceil\!\lceil H \widetilde{\rfloor\!\rfloor} = \lceil\!\lceil H \widetilde{\rfloor\!\rfloor} \widetilde{\lfloor\!\lfloor} \widetilde{E}_i^* \widetilde{\rfloor\!\rfloor}$. Also by Theorem 6.4, $A_i^* H = H\widetilde{A}_i$, so $\sum_{x' \in \lceil\!\lceil x \rceil\!\rceil} A_i^* H(x', \widetilde{x}) = \sum_{x' \in \lceil\!\lceil x \rceil\!\rceil} H\widetilde{A}_i(x', \widetilde{x})$. Computing both sides of this equation gives $\lceil\!\lceil A_i^* \rceil\!\rceil \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} = \lceil\!\lceil H \widetilde{\rfloor\!\rfloor} \widetilde{\lfloor\!\lfloor} \widetilde{A}_i \widetilde{\rfloor\!\rfloor}$. As in (i), Lemma 6.5 implies that ${}^{\mathrm{t}}\lceil\!\lceil A_i \rceil\!\rceil \lceil\!\lceil H \widetilde{\rfloor\!\rfloor} = \lceil\!\lceil H \widetilde{\rfloor\!\rfloor} \widetilde{\lfloor\!\lfloor} \widetilde{A}_i^* \widetilde{\rfloor\!\rfloor}$ and $\lceil\!\lceil E_i^* \rceil\!\rceil \lfloor\!\lfloor H \widetilde{\rceil\!\rceil} = \lceil\!\lceil H \widetilde{\rfloor\!\rfloor}{}^{\mathrm{t}}\widetilde{\lfloor\!\lfloor} \widetilde{E}_i \widetilde{\rfloor\!\rfloor}$ for some orderings as well. The proof that $\lfloor\!\lfloor H \widetilde{\rceil\!\rceil}$ is invertible is similar to the proof that $\lceil\!\lceil H \widetilde{\rfloor\!\rfloor}$ is invertible. □

Both imprimitivity and formal duality can be developed at the level of C-algebras [2] as they depend only on the parameters of a Bose–Mesner algebra. We expect that most of our results can generalized to a similar level in the abstract Terwilliger algebra of [14] which is defined using a dual pair of C-algebras.

8. Self-duality and imprimitivity

We briefly consider formal and hyper-self-duality. These cases proceeds as expected, so we omit proofs. We consider a formally self-dual Bose–Mesner algebra $\mathscr{M}$ which is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$. We write $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor_{\mathscr{I}}$ and $\lceil\!\lceil \mathscr{M} \rceil\!\rceil_{\mathscr{D}}$ to denote the block and quotient Bose–Mesner algebras of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$. It turns out that $\mathscr{M}$ is also imprimitive with respect to $(\mathscr{D}, \mathscr{I})$, so there are block and quotient Bose–Mesner algebras $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor_{\mathscr{D}}$ and $\lceil\!\lceil \mathscr{M} \rceil\!\rceil_{\mathscr{I}}$ of $\mathscr{M}$ with respect to $(\mathscr{D}, \mathscr{I})$. For simplicity we shall only consider one subconstituent algebra–in principle we could use two distinct base points.

Theorem 8.1. *Let $\mathscr{M}$ denote a Bose–Mesner algebra on X, and assume that $\Psi : \mathscr{M} \to \mathscr{M}$ is a formal duality of $\mathscr{M}$. Suppose $\mathscr{M}$ is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$. Then the following hold.*

(i) *$\mathscr{M}$ is imprimitive with respect to $(\mathscr{D}, \mathscr{I})$.*

(ii) *Let $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor_{\mathscr{I}}$ denote any block Bose–Mesner algebra of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$, and let $\lceil\!\lceil \mathscr{M} \rceil\!\rceil_{\mathscr{I}}$ denote the quotient Bose–Mesner algebra of $\mathscr{M}$ with respect to $(\mathscr{D}, \mathscr{I})$. Then the linear map $\lfloor\!\lfloor \Psi \rceil\!\rceil : \lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor_{\mathscr{I}} \to \lceil\!\lceil \mathscr{M} \rceil\!\rceil_{\mathscr{I}}$ satisfying $\lfloor\!\lfloor \Psi \rceil\!\rceil(\lfloor\!\lfloor A_i \rfloor\!\rfloor_{\mathscr{I}}) = n^{\mathrm{t}}\lceil\!\lceil E_i \rceil\!\rceil_{\mathscr{I}}$ $(i \in \mathscr{I})$is a formal duality, where $\{A_i\}_{i \in \mathscr{I}}$ and $\{E_i\}_{i \in \mathscr{I}}$ are in standard orders for Ψ.*

(iii) *Let $\lceil\!\lceil \mathscr{M} \rceil\!\rceil_{\mathscr{D}}$ denote the quotient Bose–Mesner algebra of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$, and let $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor_{\mathscr{D}}$ denote any block Bose–Mesner algebra of $\mathscr{M}$ with respect to $(\mathscr{D}, \mathscr{I})$. Then the linear map $\lceil\!\lceil \Psi \rfloor\!\rfloor : \lceil\!\lceil \mathscr{M} \rceil\!\rceil_{\mathscr{D}} \to \lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor_{\mathscr{D}}$ satisfying*

$\lceil\!\lceil \Psi \rfloor\!\rfloor(\lceil\!\lceil E_i \rceil\!\rceil_{\mathscr{D}}) = \lfloor\!\lfloor A_i \rfloor\!\rfloor_{\mathscr{D}}$ $(i \in \mathscr{D})$ is a formal duality, where $\{A_i\}_{i\in\mathscr{D}}$ and $\{E_i\}_{i\in\mathscr{D}}$ are in standard orders for Ψ.

Theorem 8.2. *Let $\mathscr{M}$ denote a Bose–Mesner algebra on X. Fix $p \in X$, and let $\mathscr{T}$ denote the subconstituent algebra of $\mathscr{M}$ with respect to p. Suppose that $\psi : \mathscr{T} \to \mathscr{T}$ is a hyper-self-duality. Further suppose that $\mathscr{M}$ is imprimitive with respect to $(\mathscr{I}, \mathscr{D})$ (so $\widetilde{\mathscr{M}}$ is imprimitive with respect to $(\mathscr{D}, \mathscr{I})$). Then the following hold.*

(i) *Let $\lfloor\!\lfloor \mathscr{T} \rfloor\!\rfloor_{\mathscr{I}}$ denote the subconstituent algebra with respect to p of the $\lfloor\!\lfloor p \rfloor\!\rfloor_{\mathscr{I}}$-block Bose–Mesner algebra $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor_{\mathscr{I}}$ of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$. Let $\lceil\!\lceil \mathscr{T} \rceil\!\rceil_{\mathscr{I}}$ denote the subconstituent algebra with respect to $\lceil\!\lceil p \rceil\!\rceil_{\mathscr{I}}$ of the quotient Bose–Mesner algebra $\lceil\!\lceil \mathscr{M} \rceil\!\rceil_{\mathscr{I}}$* of *$\mathscr{M}$ with respect to $(\mathscr{D}, \mathscr{I})$. Then there is a hyper-duality $\lfloor\!\lfloor \psi \rceil\!\rceil : \lfloor\!\lfloor \mathscr{T} \rfloor\!\rfloor_{\mathscr{I}} \to \lceil\!\lceil \mathscr{T} \rceil\!\rceil_{\mathscr{I}}$ satisfying $\lfloor\!\lfloor \psi \rceil\!\rceil({}^{\mathrm{t}}\lfloor\!\lfloor A_i \rfloor\!\rfloor_{\mathscr{I}}) = \lceil\!\lceil A_i^* \rceil\!\rceil_{\mathscr{I}}$, $\lfloor\!\lfloor \psi \rceil\!\rceil(\lfloor\!\lfloor E_i^* \rfloor\!\rfloor_{\mathscr{I}}) = {}^{\mathrm{t}}\lceil\!\lceil E_i \rceil\!\rceil_{\mathscr{I}}$ $(i \in \mathscr{I})$, where $\{A_i\}_{i\in\mathscr{I}}$ and $\{\widetilde{E}_i\}_{i\in\mathscr{I}}$ are in standard orders for ψ.*

(ii) *Let $\lceil\!\lceil \mathscr{T} \rceil\!\rceil_{\mathscr{D}}$ denote the subconstituent algebra with respect to $\lceil\!\lceil p \rceil\!\rceil_{\mathscr{D}}$ of the quotient Bose–Mesner algebra $\lceil\!\lceil \mathscr{M} \rceil\!\rceil_{\mathscr{D}}$ of $\mathscr{M}$ with respect to $(\mathscr{I}, \mathscr{D})$. Let $\lfloor\!\lfloor \mathscr{T} \rfloor\!\rfloor_{\mathscr{D}}$ denote the subconstituent algebra with respect to p of the $\lfloor\!\lfloor p \rfloor\!\rfloor_{\mathscr{D}}$-block Bose–Mesner algebra $\lfloor\!\lfloor \mathscr{M} \rfloor\!\rfloor_{\mathscr{D}}$ of $\mathscr{M}$ with respect to $(\mathscr{D}, \mathscr{I})$. Then there is a hyper-duality $\lceil\!\lceil \psi \rfloor\!\rfloor : \lceil\!\lceil \mathscr{T} \rceil\!\rceil_{\mathscr{D}} \to \lfloor\!\lfloor \mathscr{T} \rfloor\!\rfloor_{\mathscr{D}}$ satisfying $\lceil\!\lceil \psi \rfloor\!\rfloor(\lceil\!\lceil E_i \rceil\!\rceil_{\mathscr{D}}) = \lfloor\!\lfloor E_i^* \rfloor\!\rfloor_{\mathscr{D}}$, $\lceil\!\lceil \psi \rfloor\!\rfloor(\lceil\!\lceil A_i^* \rceil\!\rceil_{\mathscr{D}}) = \lfloor\!\lfloor A_i \rfloor\!\rfloor_{\mathscr{D}}$ $(i \in \mathscr{D})$, where $\{E_i\}_{i\in\mathscr{D}}$ and $\{\widetilde{A}_i\}_{i\in\mathscr{D}}$ are in standard orders for ψ.*

Theorem 8.3. *With reference to Theorem* 8.2, *suppose that the matrix $H \in \mathbb{M}_X$ represents the hyper-self-duality $\psi : \mathscr{T} \to \mathscr{T}$. Then the following hold.*

(i) *Let $\lfloor\!\lfloor H \rceil\!\rceil \in \mathbb{M}_{\lfloor\!\lfloor p \rfloor\!\rfloor_{\mathscr{I}}, \lceil\!\lceil X \rceil\!\rceil_{\mathscr{I}}}$ have $(x, \lceil\!\lceil y \rceil\!\rceil_{\mathscr{I}})$-entry*

$$\lfloor\!\lfloor H \rceil\!\rceil(x, \lceil\!\lceil y \rceil\!\rceil_{\mathscr{I}}) = \sum_{y' \in \lceil y \rceil_{\mathscr{I}}} H(x, y') \quad (x \in \lfloor\!\lfloor p \rfloor\!\rfloor, \lceil\!\lceil y \rceil\!\rceil_{\mathscr{I}} \in \lceil\!\lceil X \rceil\!\rceil_{\mathscr{I}}).$$

Then $\lfloor\!\lfloor H \rceil\!\rceil$ represents the hyper-duality $\lfloor\!\lfloor \psi \rceil\!\rceil$ of Theorem 8.2(i).

(ii) *Let $\lceil\!\lceil H \rfloor\!\rfloor \in \mathbb{M}_{\lceil\!\lceil X \rceil\!\rceil_{\mathscr{D}}, \lfloor\!\lfloor p \rfloor\!\rfloor_{\mathscr{D}}}$ have $(\lceil\!\lceil x \rceil\!\rceil_{\mathscr{D}}, y)$-entry*

$$\lceil\!\lceil H \rfloor\!\rfloor(\lceil\!\lceil x \rceil\!\rceil_{\mathscr{D}}, y) = \sum_{x' \in \lceil\!\lceil x \rceil\!\rceil_{\mathscr{D}}} H(x', y) \quad (\lceil\!\lceil x \rceil\!\rceil_{\mathscr{D}} \in \lceil\!\lceil X \rceil\!\rceil_{\mathscr{D}}, y \in \lfloor\!\lfloor p \rfloor\!\rfloor_{\mathscr{D}}).$$

Then $\lceil\!\lceil H \rfloor\!\rfloor$ represents the hyper-duality $\lceil\!\lceil \psi \rfloor\!\rfloor$ of Theorem 8.2(ii).

9. Example

The reader may find it illustrative to fill in the details of the following example associated with the Hamming cubes. The reader is referred to [3] for background material. For example, we shall implicitly use facts concerning *P*- and *Q*-polynomial Bose–Mesner algebras.

Let d denote a positive integer, and let X denote the set of all d-tuples of zeros and ones. The Hamming d-cube Q_d is the distance-regular graph with vertex set X and two vectors adjacent if they differ in exactly one coordinate. We consider hyper-duality in the subconstituent algebra of Q_d with respect to the zero vector. There is no loss in choosing this base point as the d-cube is vertex-transitive. The subconstituent algebra of Q_d is studied in [15]. The adjacency matrix A, a Q-polynomial primitive idempotent E, dual adjacency matrix $A^* = \rho(2^d E)$, and dual idempotent $E^* = \rho(A)$ are given by

$$A(\vec{u}, \vec{v}) = \begin{cases} 1 & \text{if } \vec{u}, \vec{v} \text{ differ in exactly 1 entry,} \\ 0 & \text{otherwise,} \end{cases}$$

$$E(\vec{u}, \vec{v}) = 2^{-d}(d - 2j) \text{ if } \vec{u}, \vec{v} \text{ differ in exactly } j \text{ entries,}$$

$$A^*(\vec{u}, \vec{u}) = d - 2j \text{ if } \vec{u} \text{ has exactly } j \text{ nonzero entries,}$$

$$E^*(\vec{u}, \vec{u}) = \begin{cases} 1 & \text{if } \vec{u} \text{ has exactly one nonzero entry,} \\ 0 & \text{otherwise} \end{cases}$$

for all $\vec{u}, \vec{v} \in X$. Let $H \in \mathbb{M}_X$ denote the matrix with $(\vec{u}, \vec{v})$-entry $H(\vec{u}, \vec{v}) = (-1)^{\vec{u}\cdot\vec{v}}$ for all $\vec{u}, \vec{v} \in X$. Then H represents a hyper-self-duality of Q_d:

$$ {}^{\mathrm{t}}AH = HA^*, \quad A^*H = HA, \quad EH = HE^*, \quad E^*H = H\,{}^{\mathrm{t}}E \quad (0 \leqslant i \leqslant d). $$

In particular, the Bose–Mesner algebra of Q_d is representably hyper-self-dual.

The Hamming d-cube is imprimitive (it is both bipartite and antipodal). The restriction of the adjacency relation of Q_d to either block of the bipartition of vertices induces an isomorphic copy of the halved d-cube $\frac{1}{2}Q_d$. The identification of antipodal vertices of Q_d produces the folded d-cube $\overline{Q_d}$. We now describe the hyper-duality of $\frac{1}{2}Q_d$ and $\overline{Q_d}$ following the development of Section 8. We begin by describing generators for the subconstituent algebras of $\frac{1}{2}Q_d$ and $\overline{Q_d}$.

Let $\llbracket X \rrbracket = \{\vec{u} \in X \mid \vec{u}$ has an even number of nonzero entries$\}$, a cell in the bipartition of Q_d. The halved d-cube $\frac{1}{2}Q_d$ is the distance-regular graph with vertex set $\llbracket X \rrbracket$ and two vectors adjacent if they differ in exactly two coordinates. Consider the subconstituent algebra of $\frac{1}{2}Q_d$ with respect to the zero vector. The adjacency matrix $\llbracket A \rrbracket$, Q-polynomial primitive idempotent $\llbracket E \rrbracket$, dual adjacency matrix $\llbracket A^* \rrbracket = \rho(2^{d-1}\llbracket E \rrbracket)$, and dual idempotent $\llbracket E^* \rrbracket = \rho(\llbracket A \rrbracket)$ are given by

$$ \llbracket A \rrbracket(\vec{u}, \vec{v}) = \begin{cases} 1 & \text{if } \vec{u}, \vec{v} \text{ differ in exactly 2 entries,} \\ 0 & \text{otherwise,} \end{cases} $$

$$ \llbracket E \rrbracket(\vec{u}, \vec{v}) = 2^{1-d}(d - 2j) \text{ if } \vec{u}, \vec{v} \text{ differ in exactly } j \text{ entries,} $$

$$ \llbracket A^* \rrbracket(\vec{u}, \vec{u}) = d - 2j \text{ if } \vec{u} \text{ has exactly } j \text{ nonzero entries,} $$

$$ \llbracket E^* \rrbracket(\vec{u}, \vec{u}) = \begin{cases} 1 & \text{if } \vec{u} \text{ has exactly two nonzero entries,} \\ 0 & \text{otherwise} \end{cases} $$

for all $\vec{u}, \vec{v} \in \llbracket X \rrbracket$.

Let $\lceil\!\lceil X \rceil\!\rceil = \{\vec{u} \in X \mid$ the first coordinate of $\vec{u}$ is zero$\}$, a set of representatives of antipodal pairs of vertices in Q_d. The folded d-cube $\overline{Q_d}$ is the distance-regular graph with vertex set $\lceil\!\lceil X \rceil\!\rceil$ and two vectors adjacent if they differ in either 1 or $d-1$ coordinates. Consider the subconstituent algebra of $\overline{Q_d}$ with respect to the zero vector. The adjacency matrix $\lceil\!\lceil A \rceil\!\rceil$, Q-polynomial primitive idempotent $\lceil\!\lceil E \rceil\!\rceil$, dual adjacency matrix $\lceil\!\lceil A^* \rceil\!\rceil = \rho(2^{d-1}\lceil\!\lceil E \rceil\!\rceil)$, and dual idempotent $\lceil\!\lceil E^* \rceil\!\rceil = \rho(\lceil\!\lceil A \rceil\!\rceil)$ are given by

$$ \lceil\!\lceil A \rceil\!\rceil(\vec{u}, \vec{v}) = \begin{cases} 1 & \text{if } \vec{u}, \vec{v} \text{ differ in exactly 1 or } d-1 \text{ or entries,} \\ 0 & \text{otherwise,} \end{cases} $$

$$ \lceil\!\lceil E \rceil\!\rceil(\vec{u}, \vec{v}) = 2^{1-d}((d - 2\min(j, d-j))^2 - d)/2 \quad \text{if } \vec{u}, \vec{v} \text{ differ in exactly } j \text{ or } d-j \text{ entries,} $$

$$ \lceil\!\lceil A^* \rceil\!\rceil(\vec{u}, \vec{u}) = ((d - 2\min(j, d-j))^2 - d)/2 \quad \text{if } \vec{u} \text{ has exactly } j \text{ or } d-j \text{ nonzero entries,} $$

$$ \lceil\!\lceil E^* \rceil\!\rceil(\vec{u}, \vec{u}) = \begin{cases} 1 & \text{if } \vec{u} \text{ has exactly 1 or } d-1 \text{ nonzero entries,} \\ 0 & \text{otherwise} \end{cases} $$

for all $\vec{u}, \vec{v} \in \lceil\!\lceil X \rceil\!\rceil$.

Define $\llbracket H \rceil\!\rceil \in \mathbb{M}_{\llbracket X \rrbracket, \lceil\!\lceil X \rceil\!\rceil}$ to have entries $\llbracket H \rceil\!\rceil(\vec{u}, \vec{v}) = (-1)^{\vec{u}\cdot\vec{v}}$ $(\vec{u} \in \llbracket X \rrbracket, \vec{v} \in \lceil\!\lceil X \rceil\!\rceil)$. Then as predicted by Lemma 8.3, $\llbracket H \rceil\!\rceil$ represents a hyper-duality from $\frac{1}{2}Q_d$ to $\overline{Q_d}$:

$$ {}^{\mathrm{t}}\llbracket A \rrbracket \llbracket H \rceil\!\rceil = \llbracket H \rceil\!\rceil \lceil\!\lceil A^* \rceil\!\rceil, \quad \llbracket E \rrbracket \llbracket H \rceil\!\rceil = \llbracket H \rceil\!\rceil \lceil\!\lceil E^* \rceil\!\rceil, $$

$$ \llbracket E^* \rrbracket \llbracket H \rceil\!\rceil = \llbracket H \rceil\!\rceil\,{}^{\mathrm{t}}\lceil\!\lceil E \rceil\!\rceil, \quad \llbracket A^* \rrbracket \llbracket H \rceil\!\rceil = \llbracket H \rceil\!\rceil \lceil\!\lceil A \rceil\!\rceil. $$

When $d \neq 6$, $\overline{Q_d}$ is the unique distance-regular graph whose Bose–Mesner algebra is isomorphic to that $\overline{Q_d}$. However, there are three distance-regular graphs whose Bose–Mesner algebras are isomorphic to that of $\overline{Q_6}$ (see [3, p. 264]). In fact, their subconstituent algebras with respect to every base point are isomorphic to that of $\overline{Q_6}$ with respect to the zero vector (the tools of [8] are helpful in verifying this). Thus each is representably hyper-dual to $\frac{1}{2}Q_d$.

Suppose d is even. Then the halved d-cube $\frac{1}{2}Q_d$ is antipodal and the folded d-cube $\overline{Q_d}$ is bipartite, i.e. both are imprimitive. The folded halved d-cube is isomorphic to the halved folded d-cube, and each is formally self-dual. It can be checked as outlined below that the halved folded d-cube is representably hyper-self-dual.

Let $Y = \{\vec{u} \in X \mid$ the first entry of $\vec{u}$ is zero, $\vec{u}$ has an even number of nonzero entries$\}$. The halved folded d-cube $\frac{1}{2}\overline{Q_d}$ has vertex set Y with two vectors adjacent if they differ in 2 or $d-2$ places. Consider the subconstituent algebra of $\frac{1}{2}\overline{Q_d}$ with respect to the zero vector. The adjacency matrix B, Q-polynomial primitive idempotent F, dual adjacency matrix $B^* = \rho(2^{d-2}F)$, and dual idempotent $F^* = \rho(B)$ are given by

$$B(\vec{u},\vec{v}) = \begin{cases} 1 & \text{if } \vec{u}, \vec{v} \text{ differ in exactly 2 or } d-2 \text{ or entries,} \\ 0 & \text{otherwise,} \end{cases}$$

$$F(\vec{u},\vec{v}) = 2^{2-d}(2(d/2 - \min(j, d-j))^2 - d/2)$$
$$\text{if } \vec{u}\vec{v} \text{ differ in exactly } j \text{ or } d-j \text{ entries,}$$

$$B^*(\vec{u},\vec{u}) = (2(d/2 - \min(j, d-j))^2 - d/2)$$
$$\text{if } \vec{u} \text{ has exactly } j \text{ or } d-j \text{ nonzero entries,}$$

$$F^*(\vec{u},\vec{u}) = \begin{cases} 1 & \text{if } \vec{u} \text{ has exactly 2 or } d-2 \text{ nonzero entries,} \\ 0 & \text{otherwise} \end{cases}$$

for all $\vec{u}, \vec{v} \in Y$. Let $K \in \mathbb{M}_Y$ denote the matrix with $(\vec{u}, \vec{v})$-entry $K(\vec{u}, \vec{v}) = (-1)^{\vec{u}\cdot\vec{v}}$ for all $\vec{u}, \vec{v} \in Y$. Then K represents a hyper-self-duality of $\frac{1}{2}\overline{Q_d}$:

$${}^{\mathrm{t}}BK = KB^*, \quad FK = KF^*, \quad F^*K = K\,{}^{\mathrm{t}}F, \quad B^*K = KB.$$

Although K is not in the subconstituent algebra of $\frac{1}{2}\overline{Q_d}$, it can be shown using module theoretic arguments [6] that this hyper-self-duality can be represented by some matrix K' in the subconstituent algebra of $\frac{1}{2}\overline{Q_d}$, i.e. $\frac{1}{2}\overline{Q_d}$ is strongly hyper-self-dual.

When $d \geqslant 5$, $\frac{1}{2}\overline{Q_d}$ is the unique distance-regular graph with Bose–Mesner algebra isomorphic to that of $\frac{1}{2}\overline{Q_d}$ [19]. We have not considered distance-regular graphs with Bose–Mesner algebras isomorphic to $\frac{1}{2}\overline{Q_d}$ for smaller d with regard to hyper-duality.

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